# A Type Theory for Presheaves Over a REEDY Category

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#### Abstract

We introduce a small type theory whose models are precisely the pesheaves over a given REEDY category C with a given system of *coverings*, satisfying a certain assumption of local finiteness and presentability. Our work is directly inspired from the *Globular Type Theory* of BENJAMIN, FINSTER and MIMRAM [2], and the *Simplicial Type Theory* of RIEHL and SHULMAN [5].

# **1** Some category-theoretic definitions and results

### **Direct categories**

We first recall the notion of direct category, then we introduce some tools that we will need later on.

Definition 1.1: direct category

A category  $\mathcal{D}$  is said to be a *direct category* when the following order  $\triangleleft$  on  $\mathcal{O}b(\mathcal{D})$  is well-founded.

 $a \lhd b \iff (\exists f : a \rightarrow b, f \neq \mathsf{id}_a)$ 

#### Remark 1.2

Note that a direct category has neither non-identity isomorphisms nor non-identity endomorphisms.

**1.3 coverings.** Let C be a direct category with finite slices C/c for  $c \in Ob(C)$  (that is a *locally finite* direct category), then we let for  $c \in Ob(C)$  the *covering*  $\mathcal{F}(c)$  of c be defined as the set of non-identity morphisms  $u : b \to c$  such that u has no non-trivial factorization.

$$\mathcal{F}(c) \coloneqq \{ u : b \to c \mid f \neq \mathsf{id} \land \neg(\exists (g, v), g \neq \mathsf{id}, v \neq \mathsf{id}, u = v \circ g) \}$$

We write such a morphism  $(u : b \to c) \in \mathcal{F}(c)$  as  $u : b \to c$ . We write  $\mathcal{F}^*(c)$  for  $\mathcal{F}(c) \cup \{id_c\}$ .

Since C is assumed to have finite slices, we may observe that each non-identity morphism  $f : a \to c$  factors as

$$a \xrightarrow{g} b \xrightarrow{v} a$$

And each  $u \in \mathcal{F}(c)$  admits a unique such factorization, where v = u and g = id.

Definition 1.4: Monic direct category

A direct category C is said to be *monic* if all its morphisms are monomorphisms.

#### Remark 1.5

Notice that when C is locally finite, it equivalent to ask that the morphisms of F are monomorphisms.

Definition 1.6: two-layered boundaries

See Definition 1.19 for the more general case.

**1.7** We are now going to present three important exemples, note that you may find some further references on those on the nlab (see [1])

#### Exemple 1.8 (The category G<sub>+</sub> of globes)

We consider the following category  $G_+$ .

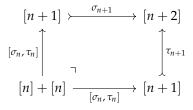
$$[0] \xrightarrow[\tau_0]{\sigma_0} [1] \xrightarrow[\tau_1]{\sigma_1} [2] \xrightarrow[\tau_2]{\sigma_2} [3] \xrightarrow[\tau_3]{\sigma_3} \cdots$$

whose set of objects is isomorphic to  $\mathbb{N}$  (we denote [*i*] the *i*-th object), generated by the morphisms  $\sigma_i$ ,  $\tau_i$ :  $[i] \rightarrow [i+1]$  subject to the *coglobular relations*:

 $(i \in \mathbb{N}) \qquad \qquad \sigma_{i+1} \circ \sigma_i = \tau_{i+1} \circ \sigma_i \qquad \qquad \sigma_{i+1} \circ \tau_i = \tau_{i+1} \circ \tau_i$ 

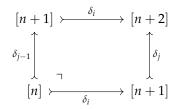
The maps  $\sigma_i$  ( $i \in \mathbb{N}$ ) are called the *cosources* and the  $\tau_i$  ( $i \in \mathbb{N}$ ) the *cotargets*.

The order  $\triangleleft$  of Definition 1.1 is isomorphic to  $\omega$ , so  $\mathbf{G}_+$  is a direct category. It is locally finite and monic, its coverings are given by  $\mathcal{F}([0]) = \emptyset$  and  $\mathcal{F}([n+1]) = \{\sigma_n, \tau_n\}$ . And it has two-layered boundaries, indeed, the pullback of  $\sigma_0$ ,  $\tau_0 : [0] \rightrightarrows [1]$  is the empty sum, and the pullback of  $\sigma_{n+1}$ ,  $\tau_{n+1} : [n+1] \rightrightarrows [n+2]$  is given by [n] + [n].



#### Exemple 1.9 (The category $\Delta_+$ of simplices)

We consider the category  $\Delta_+$  whose objects are the non-empty finite ordered sets  $[i] = \{0 < 1 < \cdots < i\}$ and whose maps are the increasing maps, called the *cofaces*. As for the category **G**, the order  $\triangleleft$  of Definition 1.1 is isomorphic to  $\omega$ , so  $\Delta_+$  is a direct category. It is locally finite and monic, its coverings are given by  $\mathcal{F}([0]) = \emptyset$  and  $\mathcal{F}([n+1]) = \{\delta_i\}_{0 \le i \le n+1}$  where  $d_i$  is the only increasing map  $[n] \rightarrow [n+1]$  such that  $i \notin \delta_i([n])$ . And it has two-layered boundaries, indeed, the pullback of  $\delta_0$ ,  $\delta_1 : [0] \rightrightarrows [1]$  is the empty sum, and the pullback of  $\delta_i$ ,  $\delta_j : [n+1] \rightrightarrows [n+2]$  (i < j) is given by [n].



#### Exemple 1.10 (The category $\square_+$ of cubes)

We consider the category  $\square_+$  whose objects are the sets  $[i] = \{0, 1\}^i$  for  $i \ge 0$ . And whose maps are the functions  $[i] \rightarrow [j]$  which inserts 0 or 1 along a tuple. That is, maps

$$\delta_{k_1,\varepsilon_1,\cdots,k_{j-i},\varepsilon_{j-i}}:[i]\longrightarrow [j]$$

described as

$$[i] \simeq [1]^i \to [1]^{k_1-1} \times \{\varepsilon_1\} \times [1]^{k_2-k_1-1} \times \cdots \times \{\varepsilon_{j-i}\} \times [1]^{j-k_{j-i}} \hookrightarrow [1]^j \simeq [j] \quad .$$

Once again, the order  $\triangleleft$  is isomorphic to  $\omega$ , making  $\square_+$  a direct category. It is also locally finite and monic, its coverings are given by  $\mathcal{F}([0]) = \emptyset$  and  $\mathcal{F}([n+1]) = \{\delta_{i,\varepsilon}\}_{0 \le i \le n+1, 0 \le \varepsilon \le 1}$ . Where

$$\begin{array}{cccc} \delta_{i,\varepsilon}: & [n] & \longrightarrow & [n+1] \\ & (x_1,\cdots,x_n) & \longmapsto & (x_1,\cdots,x_{i-1},\varepsilon,x_i,\cdots,x_n) \end{array}$$

For two maps  $\delta_{i,0} : [n] \to [n+1]$  and  $\delta_{i,1} : [n] \to [n+1]$ , their pullback in  $\square_+$  is the empty presheaf. And for any two maps  $\delta_{i,\varepsilon} : [n+1] \to [n+2]$  and  $\delta_{j,\eta} : [n+1] \to [n+2]$  where i < j, their pullback is given as follows:

$$\begin{array}{c} [n+1] \xrightarrow{\delta_{i,\varepsilon}} [n+2] \\ \delta_{j-1,\eta} & \qquad & \uparrow \\ \delta_{j,\eta} & \qquad & \uparrow \\ [n] \xrightarrow{\neg} & \qquad & \uparrow \\ \delta_{i,\varepsilon} & \qquad & [n+1] \end{array}$$

Hence  $\square_+$  has two-layered boundaries.

Definition 1.11: Boundaries

Let C be a direct category and  $c \in Ob(C)$  an object. The *boundary* of c, denoted  $\partial c$ , is the presheaf on C defined by  $\partial c(b) = \{f : b \to c \mid f \neq id\}$ .

Proposition 1.12: Decomposition of boundaries

See Proposition 1.18 for the more general case.

Definition 1.13 :  $\hat{C}_f$ 

Let C be a category. We let  $\hat{C}_f$  denotes the category of finite colimits of representable presheaves over the category C. We call  $\hat{C}_f$  the category of *finitely generated* presheaves.

# Remark 1.14

Note that  $\hat{C}_f$  is a full subcategory of  $\hat{C}$  which contains the boundaries. When C is a locally finite and direct category, since any representable presheaf is finite and any finite presheaf may be expressed as a finite colimit of representable presheaves,  $\hat{C}_f$  may alternatively be described as the category of finite presheaves over C.

# **REEDY categories**

We now get to the more general case of REEDY categories, introduce the notion of *eleguant* REEDY category and study some of their properties.

Definition 1.15 : REEDY category

A REEDY category C consist of the following data.

- A category C
- Two *wide* subcategories  $C_+$  and  $C_-$ , where *wide* means "with the same objects than C".
- A *degree* function deg :  $Ob(C) \rightarrow \alpha$  for some ordinal  $\alpha$ .

Such that

- Every morphism of  $C_-$  (resp.  $C_+$ ) lowers (resp. increases) the degree.
- $(\mathcal{M}or(\mathcal{C}_{-}), \mathcal{M}or(\mathcal{C}_{+}))$  is a *strict factorization system*, that is: every map f factors uniquely as  $f_{+} \circ f_{-}$  where  $f_{-} \in \mathcal{M}or(\mathcal{C}_{-})$  and  $f_{+} \in \mathcal{M}or(\mathcal{C}_{+})$ .

# Remark 1.16

From those properties, one see that  $C_+ \cap C_-$  contains exactly the identities. Moreover, the category  $C_+$  is always a direct category.

Definition 1.17: Boundaries

Let C be a REEDY category and  $c \in Ob(C)$  an object. The *boundary* of c, denoted  $\partial c$ , is the presheaf on C defined by  $\partial c(b) = \{f : b \to c \mid f \text{ factors through } \mathcal{F}(c)\}.$ 

Proposition 1.18: Decomposition of boundaries

Let  $c \in Ob(C)$  an object of a REEDY category, and let  $(a(u, v), p_1, p_2)$  be the choice of a pullback in  $\hat{C}$  for any two maps  $u \neq v \in \mathcal{F}(c)$ . Then  $\partial c$  may be seen as the colimit

$$\partial c = \operatorname{colim} \left( \mathcal{D}_c \to \hat{\mathcal{C}} \right)$$

Where  $\mathcal{D}_c$  is the category whose objetcs are

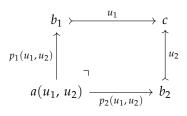
$$\mathcal{O}b(\mathcal{D}_c) = \{b_u\}_{u \in \mathcal{F}(c)} \cup \{a(u, v)\}_{u \neq v \in \mathcal{F}(c)}$$

and the arrows are the legs  $p_1$ ,  $p_2$  of each pullback. The structural maps of the colimit being given by the  $u : b_u \to \partial c$  obtained by the YONEDA lemma.

Definition 1.19: two-layered boundaries

A locally finite REEDY category C will be said to have *two-layered boundaries* if for any two distinct  $u_1, u_2 \in \mathcal{F}(c)$ , the pullback of  $u_1 : b_1 \to c$  and  $u_2 : b_2 \to c$  (well defined in  $\hat{C}$ ) decomposes as a (finite) coproduct of representable presheaves, and the projections  $p_1$ ,  $p_2$  of this pullback decompose accordingly as morphisms in  $\mathcal{F}$ .

That is, there is a finite set  $\mathcal{J}_{u_1,u_2}$  such that the coproduct  $a(u_1, u_2)$  of the  $a_j(u_1, u_2)$  ( $j \in \mathcal{J}_{u_1,u_2}$ ) in  $\hat{\mathcal{C}}$  is the aforementioned pullback. Then, writing  $p_{i,j}(u_1, u_2)$  for the *j*-th component of the projection  $p_i(u_1, u_2) : a(u_1, u_2) \to b_i$ , we have  $p_{i,j}(u_1, u_2) \in \mathcal{F}(b_i)$  for every *i*, *j*.



#### Remark 1.20

Notice that such a pullback is always a finite coproduct of representable presheaves, because it may be expressed as a weighted colimit of such, with a weight whose category of element is finite. Moreover, if it decomposes as a coproduct of representable presheaves, then this decomposition is unique, hence canonical. So having two-layered boundaries is a property, and not an additional structure on a REEDY category C.

#### Definition 1.21: Degenerated cell

Let  $X \in \hat{C}$  where C is a REEDY category, we say that a *cell* x of X (that is an element  $x \in X(c)$  for some  $c \in Ob(C)$ ) is *degenerated* iff it may be written  $p^*y$  for some other cell y and  $p \in C_-$ . A cell is said to be *non-degenerated* iff it is not degenerated.

Definition 1.22 : Eleguant REEDY category

A REEDY category C is said to be *eleguant* if every cell of any presheaf  $X \in \hat{C}$  may be written as  $p^*x$  for a unique pair (p, x) of a map  $p \in Mor(C_-)$  and a cell x which is non-degenerated.

# Exemple 1.23 (The category G of reflexive globes)

The category **G** has the same objects as **G**<sub>+</sub> (Exemple 1.8). The degree function is given by deg([*n*]) = *n*, **G**<sub>+</sub> is the category of Exemple 1.8, and **G**<sub>-</sub> is generated by the maps  $\iota : [n + 1] \rightarrow [n]$  subject to the relations  $\iota \circ \sigma = id$  and  $\iota \circ \tau = id$ . One may check that  $\mathcal{G}$  is an eleguant REEDY category with two-layered boundaries (given as for  $\mathcal{G}_+$ ).

#### Exemple 1.24 (The category $\Delta$ of simplices with degeneracies)

The category  $\Delta$  has the same objects as ffi<sub>+</sub> (Exemple 1.9). The degree function is given by deg([*n*]) = *n*,  $\Delta$ <sub>+</sub>

is the category of Exemple 1.9, and  $\Delta_{-}$  is generated by the maps  $s_i : [n + 1] \rightarrow [n] \ (0 \le i \le n)$  subject to the relations

• 
$$\sigma_i \circ \sigma_i = \sigma_i \circ \sigma_{i+1}$$
 when  $i \leq j$ 

• 
$$\sigma_j \circ \delta_i = \begin{cases} \delta_i \circ \sigma_{j-1} & \text{if } i < j. \\ \text{id} & \text{if } i \in \{j, j+1\} \\ \delta_{i-1} \circ \sigma_j & \text{if } i > j+1. \end{cases}$$

One may check that  $\Delta$  is an eleguant REEDY category with two-layered boundaries (given as for  $\Delta_+$ ).

#### Exemple 1.25 (The category of cubes with degeneracies)

The category  $\Box$  has the same objects as  $\Box_+$  (Exemple 1.10). The degree function is given by deg([n]) = n,  $\Box_+$  is the category of Exemple 1.10, and  $\Box_-$  is generated by the maps  $s_i : [n+1] \rightarrow [n]$  ( $0 \le i \le n$ ) subject to the relations

• 
$$\sigma_j \circ \sigma_i = \sigma_i \circ \sigma_{j+1}$$
 when  $i \le j$   
•  $\sigma_j \circ \delta_i = \sigma_j \circ \delta_{i,\varepsilon} = \begin{cases} \delta_{i,\varepsilon} \circ \sigma_{j-1} & \text{if } i < j \\ \text{id} & \text{if } i = j \\ \delta_{i-1,\varepsilon} \circ \sigma_i & \text{if } i > j \end{cases}$ 

(Please refer to [3] for a more precise presentation of the category of cubes.) One may check that  $\square$  is an eleguant REEDY category with two-layered boundaries (given as for  $\square_+$ ).

**1.26** In the following, we fix a locally finite and eleguant REEDY category C indexed over  $\omega$ . We will characterize the category  $\hat{C}_f$  of its finitely generated presheaves.

Lemma 1.27

If X is a representable presheaf over C, then it has a finite number of non-degenerated cells.

*Proof.* Suppose  $X = \hom(-, c)$ . Then a cell  $x : b \to c$  is non-degenerated iff it does not factor as  $y \circ p$  for some  $p : b \to a$  in  $C_-$  a non-identity. By the factorization property of REEDY categories, this happens iff  $x \in C_+$ . Howether, because  $C_+$  is direct and locally finite, the set  $\bigsqcup_{b \in Ob(C)}(\hom_{C_+}(b, c))$  is finite. Whence the result.

Lemma 1.28

Let  $X \in \hat{C}_f$ , then X admits a finite set of non-degenerated elements  $\{x_i\}_{1 \le i \le n}$ . Moreover, each element  $y \in X$  is a unique degeneracy of a unique  $x_i$ . We call the  $x_i$ 's the *generators* of X.

*Proof.* Assuming the first statement to be proven, the second one is by definition of an eleguant Reedy category. As to the first one, let  $F : \mathcal{I} \to \mathcal{C}$  be a finite diagram in  $\mathcal{C}$ , seen as a diagram in  $\hat{\mathcal{C}}$ . Suppose that X is a colimit of F, then for all  $c \in \mathcal{Ob}(\mathcal{C})$ ,  $X_c$  may be expressed as the quotient  $\bigsqcup_{i \in \mathcal{Ob}(\mathcal{I})} F(i)_c / \sim$  where  $\sim$  is the identification  $x \sim F(\alpha)(x)$  for every  $\alpha \in \mathcal{M}or(\mathcal{I})$ .

Notice that the set  $X_{nd}$  of non-degenerated cells in X is included in the set  $[\bigsqcup_{i \in Ob(\mathcal{I})} F(i)_{nd}]$  of classes of non-degenerated cells of the F(i)'s.

We then conclude that it is finite using Lemma 1.27 and the finiteness of  $\mathcal{I}$ .

Theorem 1.29

A presheaf  $X \in \hat{C}$  is finitely generated iff its set of non-degenerated cells is finite.

Proof. Lemma 1.28 gives the first implication. We now see the converse one.

Suppose that *X* has a finite set of non-degenerated cells  $X_{nd} = \{x_i\}_{1 \le i \le n}$ . Let  $\mathcal{P}_X$  denotes the full subcategory of  $\int X$  whose objects are the  $f^*x_i$  for some *i* and  $f \in \mathcal{C}_+$ . For  $h \in \mathcal{M}or(\mathcal{C})$ , there is a morphism  $h : f^*x_i \to g^*x_j$  in  $\mathcal{P}_X$  iff  $h^*g^*x_j = f^*x_i$ . Our assumptions on  $\mathcal{C}$  ensures that  $\mathcal{P}_X$  is a finite category. Let  $\pi : \mathcal{P} \to \mathcal{C}$  be the canonical projection, sending  $f^*x_i \in X_b$  to *b*, and a morphism *h* to itself. Then the YONEDA

embedding  $Y : C \to \hat{C}$  gives a cocone  $\pi \Rightarrow \Delta_X$ , where  $\Delta_X$  denotes the constant functor of value X. We now check that this cocone is universal.

Let  $\varphi : \pi \Rightarrow Y$  for some  $Y \in \hat{C}$ , if  $\varphi$  factors through  $\psi : X \to Y$ , then  $\psi(x_i) = \varphi(x_i)$  for all *i*. Using the YONEDA embedding, we may see  $\varphi(z)$  as an element in  $Y_b$  when  $z \in X_b$ . Hence,  $\psi$  is entirely and well-defined as a function, by  $\psi(f^*x_i) = f^*\varphi(x_i)$  for any *i* and  $f \in C_-$ .

Now, we check the naturality of  $\psi$ . First consider some  $x_i \in X_{nd}$  and some  $f \in Mor(\mathcal{C})$  such that  $f^*x_i$  is well defined. We write  $f = f_+ \circ f_-$  for  $f_+ \in \mathcal{C}_+$  and  $f_- \in \mathcal{C}_-$ , then we check  $\psi(f^*x_i) = f^*\psi(x_i)$ . The cell  $f_+^*x_i$  may be written  $p^*x_j$  for some j and  $p \in \mathcal{C}_-$ . By definition of  $\mathcal{P}_X$ , there is an arrow  $p : g_+^*x_i \to x_j$  in  $\mathcal{P}_X$ . Hence,  $\varphi(f_+^*x_i) = p^*\varphi(x_j)$ . Moreover,  $f^*x_i = f_-^*p^*x_j$ , then

$$\psi(f^*x_i) = \psi(f^*_- p^*x_j) = f^*_- p^*\varphi(x_j) = f^*_-\varphi(f^*_+x_i) = f^*_- f^*_+\varphi(x_i) = f^*_+\psi(x_i)$$

Let *z* be any cell of *X*, written  $z = f^*x_i$  for some *i* and  $f \in C_-$ . Let  $g \in Mor(C)$  such that  $g^*z$  makes sense. Then we shall check that  $\psi(g^*z) = g^*\psi(z)$ . That is,  $\psi(g^*f^*x_i) = g^*\psi(f^*x_i)$ . By the previous computation,  $\psi(f^*x_i) = f^*\psi(x_i)$  and  $\psi(g^*f^*x_i) = g^*f^*\psi(x_i)$  holds, whence the result.

# 2 Presheaves over direct categories

In this section we fix a locally finite, monic direct category C with two-layered boundaries. And we will define a type theory whose contexts are the finite presheaves on C. Its models will corresponds to the presheaves  $\hat{C}$ . In the following, we let  $\mathcal{J}_{u,v}$  and  $p_{i,j}(u, v)$  be the indexing sets and legs for the chosen pullbacks of two maps u, v of  $\mathcal{F}$ , sticking with the notations of Definition 1.19.

#### The type theory

We first define the formal system, which will constitute the type theory. We will refer to it as  $PRETTY_+$ , a short for *Presheaf Type Theory*, where the + refers to the case of direct categories. If we need to precise the direct category C we are working with, we will write  $PRETTY_C$ .

**2.1 Syntax.** We begin by defining the notions of *terms*, *types*, *contexts* and *substitutions* for our type theory. First, we assume having an infinite and well ordered set of *variables*, which we write  $(A, \leq)$ . We may take  $A = \omega$ . When considering an implementation of the theory, we shall ask for a decidable equality on A.

- A *term* (denoted t, s...) is an element of A (that is a variable).
- A *type* (denoted  $\tau, \sigma...$ ) is a pair  $\tau = (c, (t_u)_{u \in \mathcal{F}(c)})$  where  $c \in \mathcal{O}b(\mathcal{C})$  and  $t_u$  are terms. We let  $\underline{\tau} = c$  and  $\tau(u) = t_u$  for  $u \in \mathcal{F}(c)$ .
- A *context* (denoted Φ, Ψ...) is a list (x<sub>1</sub>:τ<sub>1</sub>, ··· , x<sub>n</sub>:τ<sub>n</sub>) where the x<sub>i</sub>'s are variables and the τ<sub>i</sub>'s are types. The empty context is denoted Ø.
- A *substitution* (denoted  $\alpha$ ,  $\beta$ ...) is a list  $\langle x_1 \mapsto t_1, \cdots, x_n \mapsto t_n \rangle$  where the  $x_i$ 's are variables and the  $t_i$ 's are terms. The empty substitution is denoted  $\langle \rangle$ .

2.2 Judgments. There are several kinds of derivable judgment in our theory. Informally:

- The judgment " $\Phi$  ctx" expresses that  $\Phi$  is a well-formed context.
- The judgment " $\Phi \vdash \tau$  type" expresses that the type  $\tau$  is a well-formed type in the context  $\Phi$ .
- The judgment " $\Phi \vdash t : \tau$ " expresses that *t* is a term of type  $\tau$  in the context  $\Phi$ .
- The judgment " $\Phi \vdash \alpha : \Psi$ " expresses that  $\alpha$  is a substitution of type  $\Psi$  in the context  $\Phi$  (we also say that  $\alpha$  is a substitution *from*  $\Phi$  *to*  $\Psi$ ).

**2.3 Free variables.** We define by induction on the syntax the set  $FV(x) \subseteq A$  of *free variables* for *x* a term, type, context or substitution.

- on terms (and variables):  $FV(t) = \{t\}.$
- on types:  $FV((c, (t_u)_{u \in \mathcal{F}(c)})) = \bigcup_{u \in \mathcal{F}(c)} FV(t_u).$

- on contexts:  $FV((x_1:\tau_1, \dots, x_n:\tau_n)) = \{x_i\}_{1 \le i \le n}$ .
- on substitutions:  $FV(\langle x_1 \mapsto t_1, \cdots, x_n \mapsto t_n \rangle) = \bigcup_{1 \le i \le n} FV(t_i)$ .

**2.4 Substitutions in terms and types.** We define, for a term *t* (resp. a type  $\tau$ ), the *action* of a substitution  $\alpha = \langle x_1 \mapsto t_1, \cdots, x_n \mapsto t_n \rangle$  on the term *t* (resp. the type  $\tau$ ), denoted  $t[\alpha]$  (resp.  $\tau[\alpha]$ ).

- *for t a term:*  $t[\alpha] = \begin{cases} t_i & \text{if } t = x_i \text{ for some } i. \\ t & \text{in the other cases.} \end{cases}$
- *for*  $\tau = (c, (t_u))$  *a type:*  $\tau[\alpha] = (c, (t_u[\alpha])).$

**2.5 Identities and compositions for substitutions.** For any context  $\Phi = (x_1:\tau_1, \dots, x_n:\tau_n)$ , we may define a substitution, called the *identity* substitution on  $\Phi$  as

$$\mathsf{id}_{\Phi} = \langle x_1 \mapsto x_1, \cdots, x_n \mapsto x_n \rangle.$$

and given two substitutions  $\alpha$  and  $\beta = \langle x_1 \mapsto t_1, \cdots, x_n \mapsto t_n \rangle$ , we may define their composition  $\beta \circ \alpha$  as

$$\langle x_1 \mapsto t_1[\alpha], \cdots, x_n \mapsto t_n[\alpha] \rangle$$

2.6 Inference rules. We give below the inference rules for the type theory:

$\frac{1}{\emptyset \text{ ctx}} \text{ CTX-EMP}$	$\frac{\Phi \vdash \tau \text{ type}}{(\Phi, x:\tau) \text{ ctx}} \text{ CTX-EXT}  (x \notin FV(\Phi))$	
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context rules

 $\frac{(\Phi, x:\tau) \operatorname{ctx}}{\Phi, x:\tau \vdash x:\tau} \operatorname{var} \qquad \qquad \frac{\Phi \vdash \tau_1 \operatorname{type} \quad \Phi \vdash t:\tau_2}{\Phi, x:\tau_1 \vdash t:\tau_2} \operatorname{wkg} \quad (x \notin \mathsf{FV}(t) \cup \mathsf{FV}(\Phi))$ 

typing rules

$$\frac{\Phi \operatorname{ctx} \qquad \bigwedge_{u \in \mathcal{F}(c)} \Phi \vdash t_u : \tau_u \qquad \bigwedge_{\substack{u_1 \neq u_2 \in \mathcal{F}(c) \\ j \in \mathcal{J}_{u_1, u_2}}} \tau_{u_1}(p_{1, j}) = \tau_{u_2}(p_{2, j})}{\Phi \vdash (c_\ell(t_u)_{u \in \mathcal{F}(c)}) \operatorname{type}} \operatorname{TYPE}$$

type introduction rule

$$\frac{\Phi \operatorname{ctx}}{\Phi \vdash \langle \rangle : \emptyset} \operatorname{SUB-EMP} \qquad \qquad \frac{\Phi \vdash \alpha : \Psi \quad (\Psi, \, x; \tau) \operatorname{ctx} \quad \Phi \vdash t : \tau[\alpha]}{\Phi \vdash \langle \alpha, \, x \mapsto t \rangle : (\Psi, \, x; \tau)} \operatorname{SUB-EXT}$$
substitution rules

# Some properties of PRETTY<sub>+</sub>

We now expose some properties of the theory and study its syntactic category. Since  $PRETTY_+$  is a special case of PRETTY which we will define later on, we postpone most of the proofs to the more general setting.

#### Lemma 2.7

The following properties may be shown by induction on the derivation trees.

- If " $\Phi \vdash \tau$  type" is derivable, so is " $\Phi$  ctx" and FV( $\tau$ )  $\subseteq$  FV( $\Phi$ ).
- If " $\Phi \vdash t : \tau$ " is derivable, so is " $\Phi \vdash \tau$  type" and  $\mathsf{FV}(t) \subseteq \mathsf{FV}(\Phi)$ .

- If " $\Phi \vdash \alpha : \Psi$ " is derivable, so are " $\Phi$  ctx" and " $\Psi$  ctx".
- If " $\Phi \vdash (c, (t_u)_{u \in \mathcal{F}(c)})$  type" is derivable, all terms  $t_u$  are typeable is the context  $\Phi$ .

Lemma 2.8 : Uniqueness of type

In a given context  $\Phi$ , a term *t* admits at most one type. That is there is at most one type  $\tau$  such that " $\Phi \vdash t : \tau$ " is derivable. Moreover, in this case, the pair  $(t : \tau)$  appears in the list  $\Phi$ .

*Proof.* By induction on the derivation tree.

Lemma 2.9: Uniqueness of derivations

A given judgment admits at most one derivation tree.

Proof. At most one inference rule leads to each form of judgment.

 Lemma 2.10

 The following rules are admissible.

  $\frac{\Psi \vdash \tau \text{ type } \Phi \vdash \alpha : \Psi}{\Phi \vdash \tau[\alpha] \text{ type }} \text{ sub-typ } \frac{\Psi \vdash t : \tau \Phi \vdash \alpha : \Psi}{\Phi \vdash t[\alpha] : \tau[\alpha]} \text{ sub-term}$ 
 $\frac{\Phi \vdash \alpha : \Psi \Psi \vdash \beta : \Theta}{\Phi \vdash \beta \circ \alpha : \Theta} \text{ sub-comp } \frac{\Phi \text{ ctx}}{\Phi \vdash \text{id}_{\Phi} : \Phi} \text{ sub-id}$ 

Lemma 2.11

For any term *t* or type  $\tau$ , when any of the following equation makes sense, it is satisfied.

 $t[\mathsf{id}_{\Phi}] = t \qquad \qquad t[\beta][\alpha] = t[\beta \circ \alpha]$  $\tau[\mathsf{id}_{\Phi}] = \tau \qquad \qquad \tau[\beta][\alpha] = \tau[\beta \circ \alpha]$ 

# The syntactic category

We will now define and characterise the syntactic category of PRETTY<sub>+</sub>.

Definition 2.12: Syntactic category

The syntactic category of the type theory PRETTY<sub>+</sub>, denoted  $S_{PRETTY_+}$  is defined as follows.

- It has as objects the contexts  $\Phi$  such that " $\Phi$  ctx" is derivable.
- It has as morphisms  $\alpha : \Phi \to \Psi$  the substitutions  $\alpha$  such that  $\Phi \vdash \alpha : \Psi$  is derivable.

#### Remark 2.13

Note that it is a well-defined category thanks to Lemmas 2.10 and 2.11.

**2.14 interpretation.** We will now define an interpretation [-]] of contexts, types and substitutions. This data will assemble as an equivalence of categories  $[-]] : S_{PRETY_+} \to \hat{C}_f$ .

Definition 2.15: Interpretation of contexts

Let  $\Phi$  be a context,  $\llbracket \Phi \rrbracket \in \hat{C}_f$  is defined as follows.

$$\llbracket \Phi \rrbracket_c = \{ t \in \mathcal{A} \mid \Phi \vdash t : \tau \text{ holds for some } \tau \text{ with } \underline{\tau} = c \}$$
  
=  $\{ t \in \mathsf{FV}(\Phi) \mid (t, (c, -)) \in \Phi \}$ 

And, for any  $t : \tau$  in  $\llbracket \Phi \rrbracket_c$  and  $u \in \mathcal{F}(c)$ ,  $u^*t = \tau(u)$ .

#### Definition 2.16: Interpretation of types

Let  $\tau$  be a type in some context  $\Phi$ ,

 $\llbracket \tau \rrbracket_{\Phi} : \partial \underline{\tau} \to \llbracket \Phi \rrbracket$ 

is defined such that for any  $(u : b \to \underline{\tau}) \in \mathcal{F}(\underline{\tau})$ ,  $[\![\tau]\!]_{\Phi,b}(u) = \tau(u)$ .

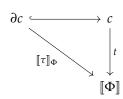
#### Remark 2.17

By factorizability of morphisms of C as maps in  $\mathcal{F}$ , the above determines completely the presheaf  $\llbracket \Phi \rrbracket$  or the transformation  $\llbracket \tau \rrbracket_{\Phi}$ . However, we shall check that both are well-defined, this is done with the following lemma.

Lemma 2.18

For any context  $\Phi$  (resp. type  $\tau$  in a context  $\Phi$ ), its interpretation  $\llbracket \Phi \rrbracket$  (resp.  $\llbracket \tau \rrbracket_{\Phi}$ ) is well-defined. Moreover, the following holds:

(i) For a context  $\Phi$  such that  $\Phi \vdash x : \tau$  and  $f : b \to \underline{\tau}$ ,  $f^*x = [\![\tau]\!]_{\Phi,b}(f)$ . That is, we have the following commutative diagram in  $\hat{C}_f$ , where  $c = \underline{\tau}$ .



(ii) For a type  $\tau = (c, (t_u)_{u \in \mathcal{F}(c)})$  in a context  $\Phi$  and  $u : b_u \to c, [\tau]_{\Phi}|_{b_u} = [\tau_u]$  where  $\Phi \vdash t_u : \tau_u$ .

Definition 2.19: Interpretation of substitutions

For any substitution  $\alpha = \langle x_1 \mapsto t_1, \cdots, x_n \mapsto t_n \rangle$  such that  $\Phi \vdash \alpha : \Psi$  is derivable, we let

 $\llbracket \alpha \rrbracket : \llbracket \Psi \rrbracket \to \llbracket \Phi \rrbracket$  be defined by  $\llbracket \alpha \rrbracket (x_i) = t_i$ 

That is, we have the equations  $[\![\alpha]\!](x) = x[\alpha]$ .

Lemma 2.20

Definition 2.19 yields a natural transformation, preserves identities and reverse compositions.

*Proof.* Every morphism  $f : z \to c$  in C factors as a finite composition of maps  $u_i$  of  $\mathcal{F}$ :

 $f=u_n\circ\cdots\circ u_1$ 

for some  $n \in \mathbb{N}$ . So it suffices to show that  $\llbracket \alpha \rrbracket$  commutes with the  $u_i$ 's.

Let  $x \in \llbracket \Psi \rrbracket_c$  for some  $c \in \mathcal{O}b(\mathcal{C})$ . Then  $\Psi \vdash x : \tau$  is derivable for some  $\tau$  with  $\underline{\tau} = c$ . Hence, using Lemma 2.10, we have  $\Phi \vdash t[\alpha] : \tau[\alpha]$  derivable, so  $t[\alpha] = \llbracket \alpha \rrbracket(t) \in \llbracket \Psi \rrbracket_c$ . Moreover, for any  $u \in \mathcal{F}(c)$ ,  $u^*(t[\alpha]) = \tau[\alpha](u) = \tau(u)[\alpha] = (u^*t)[\alpha]$ , whence  $\llbracket \alpha \rrbracket(u^*t) = u^*(\llbracket \alpha \rrbracket(t))$ . Which yields the naturality. We see by definition that identity substitution is sent to identity transformation and that composites of substitutions are sent to the reverse composites of transformations.  $\Box$ 

Definition 2.21 :  $\llbracket - \rrbracket : \mathcal{S}_{PRETTY} \to \hat{\mathcal{C}}_{f}^{op}$ 

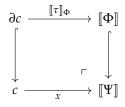
The interpretation of contexts and substitutions as given by Definitions 2.15 and 2.19 yields a contravariant functor from the syntactic category to the category of finite presheaves on C, which we denote [-] and call the *interpretation* or *semantic* functor.

Lemma 2.22

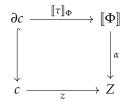
Let  $\alpha : \partial c \to \llbracket \Phi \rrbracket$  for some context  $\Phi$ , then  $\alpha = \llbracket \tau_{\alpha} \rrbracket_{\Phi}$  for some unique well-formed type  $\tau_{\alpha}$  in the context  $\Phi$ .

#### Lemma 2.23

Let  $\Phi$  be a context and  $\Psi = (\Phi, x : \tau)$  obtained by ctx-EXT. Then  $\llbracket \Psi \rrbracket$  is the following pushout:



*Proof.* The commutativity of the square above is by definition of  $\llbracket \Psi \rrbracket$ . Let *Z* be a presheaf over *C* and  $\alpha : \llbracket \Phi \rrbracket \rightarrow Z, z \in Z_c$  such that the following square commutes:



If  $\alpha$  and z factor through some  $\beta : \llbracket \Psi \rrbracket \to Z$ , then  $\beta$  is completely determined as a function by  $\beta|_{\llbracket \Phi \rrbracket} = \alpha$ and  $\beta(x) = z$ . In order to see that  $\beta$  defined as such is natural, we need to check that for any morphism  $f : a \to c$ ,  $\alpha(f^*x) = f^*z$ . Assuming  $f \neq id$ , this is given by the assumption  $\alpha \circ \llbracket \tau \rrbracket_{\Phi} = z$  together with the point (i) of Lemma 2.18.

#### Theorem 2.24

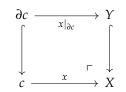
 $\llbracket - \rrbracket : S_{PRETTY_+} \longrightarrow \hat{C}_f^{op}$  is an equivalence of categories.

*Proof.* We need to check that [-] is fully faithfull and essentially surjective. We fix two contexts  $\Phi$  and  $\Psi$ .

- *faithfullness*. Let φ : [[Ψ]] → [[Φ]] be a natural transformation. Suppose φ = [[α]] for some substitution α. Then α must be of the form ⟨x ↦ φ(x)⟩<sub>x∈FV(Ψ)</sub>, whence the faithfullness.
- *fullness.* Let  $\varphi : \llbracket \Psi \rrbracket \to \llbracket \Phi \rrbracket$  be a natural transformation and let  $\alpha = \langle x \mapsto \varphi(x) \rangle_{x \in \mathsf{FV}(\Psi)}$ . For any  $t \in \llbracket \Psi \rrbracket_c, t$  is a variable in  $\mathsf{FV}(\Psi)$  according to Lemma 2.8. Hence  $\llbracket \alpha \rrbracket(t) = \varphi(t)$ , whence  $\varphi = \llbracket \alpha \rrbracket$ .
- *essential surjectivity.* Let  $X \in \hat{C}_f$ . According to Remark 1.14, X admits a finite number of elements  $x_i$   $(1 \le i \le n)$ . We proceed by induction on n.

- Suppose n = 0, then X is the empty presheaf, and is the image of the empty context.

- Suppose n > 0. Suppose  $x = x_n \in X_c$  is maximal in the sense that it may not be written as  $f^*y$  for some other cell y of X. Let  $Y := X \setminus \{x\}$ . Since x is maximal, we may check that Y is again a (finite) presheaf. There is an inclusion  $Y \hookrightarrow X$  and a map  $c \xrightarrow{x} X$  given by the YONEDA embedding. We see that those two maps exhibit X as the following pushout:



Let  $z : c \to Z$  (seen as  $z \in Z_c$ ) and  $\varphi : Y \to Z$  such that  $\varphi \circ x|_{\partial c} = z|_{\partial c}$ . If,  $\varphi$  and z factor through  $\psi : X \to Z$ , then  $\psi(x) = z$  and  $\psi|_Y = \varphi : Y \to Z$ , so  $\psi$  is completely defined as a natural transformation  $X \to Z$ .

In order to see that  $\psi$  defined as such is a well-defined natural transformation, we need to see that for any  $f : z \to c$ ,  $\psi(f^*x) = f^*\psi(x)$ . Assuming  $f \neq id$ , this equation is precisely given by the assumption  $\varphi \circ x|_{\partial c} = z|_{\partial c}$ .

Now, using the induction hypothesis with *Y* yields a context  $\Phi$  such that  $\llbracket \Phi \rrbracket \simeq Y$ , and Lemma 2.22 gives a type  $\tau$  in  $\Phi$  such that  $\llbracket \tau \rrbracket = x|_{\partial c}$ . Then Lemma 2.23 proves that  $X \simeq \llbracket \Phi, x : \tau \rrbracket$ , where  $\Phi, x : \tau$  is obtained by CTX-EXT from  $\Phi$ .

# **3** Presheaves over **REEDY** categories

**3.1** Assumptions. In this section we fix C an elegant REEDY category with two-layered boundaries, indexed over the ordinal  $\omega$ , with two wide subcategories  $C_+$  and  $C_-$  as classes of upward and downard maps. We moreover assume that :

- The category  $C_+$  is locally finite and monic.
- We are given a finite presentation P of  $C_-$  which does not contains identities.

Our aim is to define an extension of PRETTY<sub>+</sub>, which we call PRETTY (or PRETTY<sub>C</sub>), whose syntactic categories consists of finitety generated presheaves over C, and whose models are the presheaves over C. We stick to the notations  $\mathcal{F}$ ,  $\mathcal{J}_{u,v}$ ,  $p_{i,i}(u, v)$  and  $a_i(u, v)$  introduced in Section 2. For any composable pair

$$c' \xrightarrow{v \in \mathcal{F}(d)} d \xrightarrow{p \in P} c$$

such that  $p \circ v \neq id$ , we will assume the unique factorisation of  $p \circ v$  in  $(\mathcal{C}_{-}, \mathcal{C}_{+})$  to be given by

$$c' \xrightarrow{q(p,v) \in P} b \xrightarrow{w(p,v) \in \mathcal{F}(c)} c$$

# The type theory

We first define the theory PRETTY, an extension of the theory PRETTY<sub>+</sub>. If we need to precise the category C we are working with, we will write PRETTY<sub>C</sub>.

**3.2 Terms.** We consider the type theory for presheaves over  $C_+$ , which we denote PRETTY<sub>+</sub> and extend it to the type theory PRETTY by adding formal degenerescences to variables. The terms of PRETTY are given by the following grammar:

where A still denotes a denumerable set of variables as assumed in Subsection 2. We denote their set by tm.

**3.3** Syntax. As the theory PRETTY<sub>+</sub>, PRETTY has *terms*, *types*, *contexts* and *substitutions*. The terms are given by 3.3. The syntax of types, contexts, and substitutions remains unchanged from PRETTY<sub>+</sub>.

**3.4 Judgments.** In addition to the four kind of judgment introduced for PRETTY<sub>+</sub>, we add the following kinds.

- The judgment " $\Phi \vdash t \equiv u$ " expresses that in the context  $\Phi$ , the terms *t* and *u* are semantically equals.
- The judgment " $\Phi \vdash \tau \equiv \sigma$  type" expresses that in the context  $\Phi$ , the types  $\tau$  and  $\sigma$  are semantically equals.
- The judgment " $\vdash \Phi \equiv \Psi \operatorname{ctx}$ " expresses that the contexts  $\Phi$  and  $\Psi$  are semantically equals.

**3.5 Free variables.** The free variables of a term are now given by:

- $\mathsf{FV}(x) = \{x\}$  for  $x \in \mathcal{A}$ .
- $FV(p^*t) = FV\{t\}$  for *t* a term and  $p \in P$ .

The definition of free variables for types, contexts, and substitution remains unchanged from PRETTY<sub>+</sub>.

**3.6 Substitutions in terms and types.** We define the action of a substitution  $\alpha = \langle x_i \mapsto t_i \rangle_i$  in a term as follows.

• For *x* a variable:

$$x[\alpha] = \begin{cases} t_i & \text{if } x = x_i \text{ for some } i. \\ x & \text{in the other cases.} \end{cases}$$

• For *t* a term and  $p \in P$ :

$$(p^*t)[\alpha] = p^*(t[\alpha])$$

Substitutions act on types exactly as they do in PRETTY<sub>+</sub> (*c.f.* 2.4).

**3.7** Action of degeneracies on types. Let  $(p : d \to c) \in P$ , t a term, and  $\tau = (c, (t_u)_{u \in \mathcal{F}(c)})$  a type. Then we define a type  $(p, t)^* \tau = (d, (s_v)_{v \in \mathcal{F}(d)})$ , where

$$s_v = \begin{cases} t & \text{if } p \circ v = \mathsf{id}_c. \\ q(p, v)^* t_{w(p, v)} & \text{in the other cases.} \end{cases}$$

Notice that when it makes sense, the following equation holds:

$$((p, t)^* \tau)[\alpha] = (p, t)^* (\tau[\alpha])$$

3.8 Inference rules. In addition to the rules already introduced for PRETTY<sub>+</sub>, we add the following ones:

$$\frac{\Phi \vdash t : \tau \quad (p : - \to \underline{\tau}) \in P}{\Phi \vdash p^* t : (p, t)^* \tau} \text{ Dege} \qquad \qquad \frac{\Phi \vdash t : \tau \quad (p_1 \circ \dots \circ p_n) P(q_1 \circ \dots \circ q_m)}{\Phi \vdash p_n^* \cdots p_1^* t \ \equiv \ q_m^* \cdots q_1^* t} \text{ Rel}$$

degeneracy rules

$$\frac{\Phi \vdash t : \tau}{\Phi \vdash t \equiv t} \operatorname{REFL} \qquad \frac{\Phi \vdash t \equiv u}{\Phi \vdash u \equiv t} \operatorname{SYM} \qquad \frac{\Phi \vdash t \equiv u}{\Phi \vdash t \equiv v} \operatorname{TRANS}$$

$$\frac{\Phi \vdash t : \tau}{\Phi \vdash u : \tau} \qquad \Phi \vdash t \equiv u \qquad (p : - \to \underline{\tau}) \in P}{\Phi \vdash p^* t \equiv p^* u} \operatorname{EQUIV}$$

$$\frac{\Phi \vdash \tau \operatorname{type} \qquad \Phi \vdash \sigma \operatorname{type} \qquad \bigwedge_{u \in \mathcal{F}(c)} \Phi \vdash \tau(u) \equiv \sigma(u)}{\Phi \vdash \tau \equiv \sigma \operatorname{type}} \operatorname{TYPE-EQ} \qquad \underline{\tau} = \underline{\sigma} = c$$

$$\frac{\vdash \Phi \equiv \Psi \operatorname{ctx} \qquad \Phi \vdash \tau \equiv \sigma \operatorname{type}}{\vdash (\Phi, x : \tau) \equiv (\Psi, x : \sigma) \operatorname{ctx}} \operatorname{CTX-EQ} \qquad x \notin \operatorname{FV}(\Phi) \cap \operatorname{FV}(\Psi)$$
equality rules

# Some properties of PRETTY

The following lemmas may be proven by induction on the derivation trees.

Lemma 3.9

The following properties may be shown by induction on the derivation trees.

- If " $\Phi \vdash \tau$  type" is derivable, so is " $\Phi$  ctx" and  $\mathsf{FV}(\tau) \subseteq \mathsf{FV}(\Phi)$ .
- If " $\Phi \vdash t : \tau$ " is derivable, so is " $\Phi \vdash \tau$  type" and  $FV(t) \subseteq FV(\Phi)$ .
- If " $\Phi \vdash \alpha : \Psi$ " is derivable, so are " $\Phi$  ctx" and " $\Psi$  ctx".
- If " $\Phi \vdash (c, (t_u)_{u \in \mathcal{F}(c)})$  type" is derivables, all terms  $t_u$  are typeable is the context  $\Phi$ .
- If " $\Phi \vdash t \equiv s$ " is derivable, then there are types  $\tau$ ,  $\sigma$  such that  $\Phi \vdash t : \tau$ ,  $\Phi \vdash s : \tau$  and  $\Phi \vdash \tau \equiv \sigma$  holds.
- If " $\vdash \Phi \equiv \Psi$  ctx" is derivable, then a judgment " $\Phi \vdash \tau$  type" holds iff " $\Psi \vdash \tau$  type" holds. And similarly, " $\Phi \vdash t : \tau$ " holds iff " $\Psi \vdash t : \tau$ " holds.

Lemma 3.10: Uniqueness of type

In a given context  $\Phi$ , a term *t* admits at most one type up to  $\equiv$ .

Lemma 3.11

A judgment  $\Phi \vdash t : \tau$  holds iff there is a sequence of composable elements  $p_1, \dots, p_n \in P$ , a variable  $x \in FV(\Phi)$  and a type  $\sigma$  such that

 $t = p_n^* \cdots p_1^* x \qquad \tau = (p_n, p_{n-1}^* \cdots p_1^* x)^* \cdots (p_1, x)^* \sigma \qquad \text{with } (x : \sigma) \in \Phi.$ 

Lemma 3.12

 $\Phi \vdash t \equiv s$  holds iff there are sequences  $p_1, \dots, p_n \in P$  and  $q_1, \dots, q_m \in P$  and a variable  $x \in FV(\Phi)$  such that

 $t = p_n^* \cdots p_1^* x$   $s = q_m^* \cdots q_1^* x$  with  $(p_1 \circ \cdots \circ p_n) P(q_1 \circ \cdots \circ q_m)$ 

Hence we denote unambiguously  $f^*x$  for the class of  $p_n^* \cdots p_1^*x$  in  $\mathcal{T}_m/\equiv$  given by any choice of decomposition  $f = p_1 \circ \cdots \circ p_n$  with  $p_i \in P$  for all *i*'s.

# The syntactic category

**3.13** As for the direct case, the rules exposed in Lemma 2.10 are still admissible. Moreover, notice that every valid derivation tree in  $PRETTY_+$  remains valid in PRETTY, hence every context (resp. substitution) in  $PRETTY_+$  is a context (resp. substitution) in  $PRETTY_+$ .

Definition 3.14 :  $S_{PRETTY}$ 

We let  $S_{PRETTY}$  denotes the category whose objects are the well-formed contexts of PRETTY, and morphisms are the well-defined substitutions up to  $\equiv$ . According to the previous assertion, there is a functor  $S_{PRETTY_+} \rightarrow S_{PRETTY}$  which is the identity on objects and morphisms.

Definition 3.15: Interpretation of contexts

For  $\Phi$  ctx a well-formed context of PRETTY, its *interpretation* is the finitely generated presheaf  $\llbracket \Phi \rrbracket \in \hat{\mathcal{C}}_f$  defined by

 $\llbracket \Phi \rrbracket_c = \{t : \tau \mid \Phi \vdash t : \tau \text{ is provable and } \underline{\tau} = c\} / \equiv$ 

And, for any  $t : \tau$  in  $\llbracket \Phi \rrbracket_c$  and  $u \in \mathcal{F}(c)$ ,  $\llbracket \Phi \rrbracket(u)(t) = \tau(u)$ , for any  $p \in P$ ,  $\llbracket \Phi \rrbracket(p)(x) = p^*x$ .

#### Definition 3.16: Interpretation of types

For  $\Phi \vdash \tau$  type a derivable judgment, the *interpretation* of  $\tau$  is the natural transformation

 $[\![\tau]\!]_\Phi:\partial\underline{\tau}\to[\![\Phi]\!]$ 

such that for  $(u : b \to \underline{\tau}) \in \mathcal{F}(\underline{\tau})$ ,  $[\![\tau]\!]_{\Phi, b}(u) = \tau(u)$ .

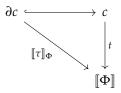
#### Remark 3.17

By factorizability of morphisms of C as maps in  $\mathcal{F}$  and P, Definition 3.15 above determines completely the presheaf  $\llbracket \Phi \rrbracket$ . On the other hand, since any element of  $\partial \underline{\tau}$  factorizes by some element  $u \in \mathcal{F}(\underline{\tau})$ , the transformation  $\llbracket \tau \rrbracket_{\Phi}$  is also fully determined by the specification of Definition 2.16. However, we shall check that both are well-defined, this is done with the following lemma.

#### Lemma 3.18

For any context  $\Phi$  (resp. type  $\tau$  in a context  $\Phi$ ), its interpretation  $\llbracket \Phi \rrbracket$  (resp.  $\llbracket \tau \rrbracket_{\Phi}$ ) is well-defined. Moreover, the following holds:

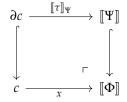
(i) For a context  $\Phi$  such that  $\Phi \vdash x : \tau$  and  $f : b \to \underline{\tau}$ ,  $f^*x = [[\tau]]_{\Phi,b}(f)$ . That is, we have the following commutative diagram in  $\hat{C}_f$ , where  $c = \underline{\tau}$ .



(ii) For a type  $\tau = (c, (t_u)_{u \in \mathcal{F}(c)})$  in a context  $\Phi$  and  $u : b_u \to c, [\tau]_{\Phi}|_{b_u} = [\tau_u]$  where  $\Phi \vdash t_u : \tau_u$ .

*Proof.* We proceed by induction on derivation trees to prove those properties.

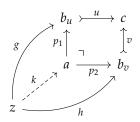
- (*well-definedness of*  $\llbracket \Phi \rrbracket$ ) Suppose that " $\Phi$  ctx" is derivable.
  - Either  $\Phi = \emptyset$  is the empty context and  $\llbracket \Phi \rrbracket = \emptyset$  is well-defined as the empty presheaf.
  - Or Φ = (Ψ, x : τ) is obtained by context extension (using CTX-EXT) of Ψ. Let c = <u>τ</u>. In this case, [[Ψ]] and [[τ]]<sub>Ψ</sub> : ∂c → [[Ψ]] are already defined. Hence [[Φ]] may be described as the following pushout.



• (*well-definedness of*  $[\![\tau]\!]_{\Phi}$ ) Suppose that " $\Phi \vdash \tau$  type" is derivable. Then the following premises of rule TYPE holds.

$$\Phi \operatorname{ctx} \qquad \bigwedge_{u \in \mathcal{F}(c)} \Phi \vdash t_u : \tau_u \qquad \bigwedge_{\substack{u_1 \neq u_2 \in \mathcal{F}(c) \\ j \in \mathcal{J}_{u_1, u_2}}} \tau_{u_1}(p_{1, j}) = \tau_{u_2}(p_{2, j})$$

Suppose given two distinct factorizations  $f = u \circ g = v \circ h$  of some element  $f \in (\partial c)_z$  where  $\underline{\tau} = c$  and  $u : b_u \rightarrow c, v : b_v \rightarrow c$ . By monicness of u, v it must be the case that  $u \neq v$ . Then the universal property of the pullback a(u, v) yield a commutative diagram in  $\hat{C}$ .



Since  $z \in C$ ,  $k : z \to a$  must factor as  $k_j : z \to a_j$  through some  $a_j \hookrightarrow a$ , for some  $j \in \mathcal{J}_{u,v}$ . Hence, the assumption  $\tau_u(p_{1,j}) = \tau_v(p_{2,j})$  allows us to write

$$g^*t_u = k_j^* p_{1,j}^* t_u = k_j^* \tau_u(p_{1,j}) = k^* \tau_v(p_{2,j}) = k_j^* p_{2,j}^* t_v = g^* t_v$$

We then see that  $[\tau]_{\Phi}$  is natural: if  $f_1 : a_1 \to c$  equals  $f_2 \circ g$  for  $f_1$  and  $f_2 : a_2 \to c$  two elements of  $\partial c$ , then picking a factorization  $f_2 = u \circ h$  for some  $u \in \mathcal{F}(c)$  yields

$$[\![\tau]\!]_{\Phi}(f_1) = [\![\tau_u]\!]_{\Phi}(g \circ h) = g^*[\![\tau_u]\!]_{\Phi}(h) = g^*[\![\tau]\!]_{\Phi}(f_2)$$

(i) Suppose that  $\Phi \vdash x : \tau$  is derivable and let  $f : b \to c$  where  $c = \underline{\tau}$ . Let  $f = u \circ g$  be a factorization where  $u \in \mathcal{F}$ . Then

$$f^*x = g^*u^*x = g^*\tau(u) = [\![\tau]\!]_{\Phi}(u \circ g) = [\![\tau]\!]_{\Phi,b}(f)$$

Where we have used the definition of  $\llbracket \Phi \rrbracket$  and the naturality of  $\llbracket \tau \rrbracket$ .

(ii) Suppose  $\tau = (c, (t_u)_{u \in \mathcal{F}(c)})$  is such that " $\Phi \vdash \tau$  type" is derivable. Let  $u : b_u \rightarrow c$  and suppose  $t_u : \tau_u$ . Then for any  $f : z \rightarrow b_u$ , we have

$$[[\tau]]_{z}(u \circ f) = f^{*}[[\tau]]_{b_{u}}(u) = f^{*}\tau(u) = [[\tau_{u}]]_{\Phi}(f)$$

Where we have used the naturality of  $[\tau]_{\Phi}$  and the point (i).

#### Definition 3.19

For any substitution  $\alpha = \langle x_1 \mapsto t_1, \cdots, x_n \mapsto t_n \rangle$  such that  $\Phi \vdash \alpha : \Psi$  is derivable, we let

 $\llbracket \alpha \rrbracket : \llbracket \Psi \rrbracket \to \llbracket \Phi \rrbracket$  be defined by  $\llbracket \alpha \rrbracket (f^* x_i) = f^* t_i$ 

That is, we have the equations  $\llbracket \alpha \rrbracket(t) = t[\alpha]$ .

#### Lemma 3.20

Definition 3.19 yields a natural transformation, preserves identities and reverse compositions.

*Proof.* Let  $\alpha$  be a substitution from  $\Phi$  to  $\Psi$ . Let  $t \in \llbracket \Psi \rrbracket_c$  for some  $c \in Ob(C)$ . Then  $\Psi \vdash t : \tau$  is derivable for some  $\tau$  with  $\underline{\tau} = c$ . Hence as mentionned in 3.13, we have  $\Phi \vdash t[\alpha] : \tau[\alpha]$  derivable, whence  $t[\alpha] = \llbracket \alpha \rrbracket(t) \in \llbracket \Psi \rrbracket_c$ . Hence  $\llbracket \alpha \rrbracket$  is well-defined.

Now we check the naturality. By assumptions on C, any morphism is a composition of maps  $u \in \mathcal{F}(c)$  for some c and  $p \in P$ . So it suffices to show that  $[\![\alpha]\!]$  commutes with the u's and the p's.

- for any  $u \in \mathcal{F}(c)$ ,  $\llbracket \Phi \rrbracket(u)(t[\alpha]) = \tau[\alpha](u) = \tau(u)[\alpha] = (\llbracket \Phi \rrbracket(u)(t))[\alpha].$
- for any  $p \in P$ ,  $\llbracket \Phi \rrbracket(p)(t[\alpha]) = p^*(t[\alpha]) = (p^*t)[\alpha] = (\llbracket \Phi \rrbracket(p)(t))[\alpha].$

Whence the naturality.

We see by definition that identity substitution is sent to identity transformation and that composites of substitutions are sent to the reverse composites of transformations.  $\Box$ 

Definition 3.21 :  $\llbracket - \rrbracket : \mathcal{S}_{PRETTY} \to \hat{\mathcal{C}}_{f}^{op}$ 

The interpretation of contexts and substitutions as given by Definitions 3.15 and 3.19 gives a contravariant functor from the syntactic category to the category of finite presheaves on C, which we denote [-] and call the *interpretation* or *semantic* functor.

#### Lemma 3.22

Let  $\alpha : \partial c \to \llbracket \Phi \rrbracket$  for some context  $\Phi$ , then  $\alpha = \llbracket \tau_{\alpha} \rrbracket$  for some unique well-formed type  $\tau_{\alpha}$  up to  $\equiv$  in the context  $\Phi$ .

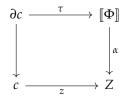
*Proof.* We proceed by induction on the object  $c \in Ob(C)$ , using the well-foundedness of  $\triangleleft$ . Using the universal property of  $\partial c$  given by Proposition 1.18,  $\alpha$  restricts to maps  $\alpha_u : b_u \to \llbracket \Phi \rrbracket$  for all  $(u : b_u \to c) \in \mathcal{F}(c)$ , which corresponds to terms  $(\alpha(u) = t_u : \tau_u) \in \llbracket \Phi \rrbracket_{b_u}$  according to Definition 3.15, for  $t_u, \tau_u$  defined up to  $\equiv$ . By Lemma 3.18,  $\llbracket \tau_u \rrbracket_{\Phi}$  is the restriction of  $t_u$  along its boundary  $\partial b_u \to \llbracket \Phi \rrbracket$  for each u. Moreover, still by definition of  $\partial c$ , those maps  $\alpha_{u_1}, \alpha_{u_2}$  (with  $u_1 \neq u_2$ ) must coincide when restricted along the legs  $p_1, p_2$  of the associated pullback. According to Definition 3.15, it gives the equations  $\tau_{u_1}(p_{1,j}) = \tau_{u_2}(p_{2,j})$  for  $j \in \mathcal{J}_{u_1,u_2}$ . Hence the type  $\tau = (c, (t_u)_{u \in \mathcal{F}(c)})$  is well formed in the context  $\Phi$ . Now, by Definition 3.16 and Lemma 3.18,  $\llbracket \tau \rrbracket_{\Phi} = \alpha$  and  $\tau$  as described above was the only possible type up to  $\equiv$ .

Lemma 3.23

Let  $\Phi$  be a context and  $\Psi = (\Phi, x : \tau)$  obtained by CTX-EXT. Then  $\llbracket \Psi \rrbracket$  is the following pushout, where  $\pi : \Psi \to \Phi$  is the canonical substitution:

$$\begin{array}{ccc} \partial c & & \llbracket \pi \rrbracket_{\Phi} & & \llbracket \Phi \rrbracket \\ & & & & & \\ \downarrow & & & & \\ c & & & & \\ c & & & & \\ \end{array} \xrightarrow[]{} r & & & \llbracket \Psi \rrbracket$$

*Proof.* The commutativity of the square above is by definition of  $\llbracket \Psi \rrbracket$ . Let *Z* be a presheaf over *C* and  $\alpha : \llbracket \Phi \rrbracket \rightarrow Z, z \in Z_c$  such that the following square commutes:



If  $\alpha$  and *z* factor through some  $\beta : \llbracket \Psi \rrbracket \to Z$ , then  $\beta$  is completely defined as a function by

- $\beta(t) = \alpha(t)$  if  $t = p^*y$  for some  $p \in C_-$  and  $y \in \mathsf{FV}(\Phi)$ .
- $\beta(p^*x) = p^*z$  for  $(p: \rightarrow c) \in \mathcal{C}_-$ .

Now, we check that  $\beta$  is well-defined as a natural transformation.

- If  $t \in \llbracket \Phi \rrbracket_b$  and  $(f : \rightarrow b) \in C$ ,  $\beta(f^*t) = \alpha(f^*t) = f^*\alpha(t) = f^*\beta(t)$ .
- If t = x and  $(f : \rightarrow c) \in C$ , such that  $f = f_+ \circ f_-$  with  $f_+ \in C_+$  and  $f_- \in C_-$ , then  $\beta(f^*x) = \beta(f^*_-f^*_+x)$ 
  - Either  $f_+ = id$ , hence  $f_- = f$  and  $\beta(f^*x) = f^*\beta(x)$  by definition
  - Or  $f_+ \neq id$ , hence  $f_-^* f_+^* x \in \llbracket \Phi \rrbracket$  and  $\beta(f_-^* f_+^* x) = \alpha(f_-^* f_+^* x) = f_-^* \alpha(f_+^* x)$ . Then  $\alpha(f_+^* x) = f_+^* z = f_+^* \beta(x)$  because the above square is commutative, whence  $\beta(f^* x) = f^* \beta(x)$ .

• If  $t = p^*x$  for some  $p \in C_-$ , then the previous point shows that for any f,  $\beta(f^*p^*x) = f^*p^*\beta(x) = f^*\beta(p^*x)$ , whence  $\beta(f^*t) = f^*\beta(t)$ .

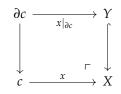
This proves the naturality condition, hence it show that  $\llbracket \Psi \rrbracket$  satisfies the universal property of the pushout.

#### Theorem 3.24

[-] is an equivalence of categories.

*Proof.* We need to check that [-] is fully faithfull and essentially surjective. We fix two contexts  $\Phi$  and  $\Psi$  of PRETTY.

- *faithfullness*. Let  $\varphi : \llbracket \Psi \rrbracket \to \llbracket \Phi \rrbracket$  be a natural transformation. Suppose  $\varphi = \llbracket \alpha \rrbracket$  for some substitution  $\alpha$ . Then  $\alpha$  must be of the form  $\langle x \mapsto \varphi(x) \rangle_{x \in \mathsf{FV}(\Psi)}$  (which is well-defined, up to  $\equiv$ ), whence the faithfullness.
- *fullness.* Let  $\varphi : \llbracket \Psi \rrbracket \to \llbracket \Phi \rrbracket$  be a natural transformation and let  $\alpha = \langle x \mapsto \varphi(x) \rangle_{x \in \mathsf{FV}(\Psi)}$ . For any  $t \in \llbracket \Psi \rrbracket_c$ , *t* is of the form  $f^*x$  for some  $f \in \mathcal{C}_-$ . By naturality,  $\varphi(t) = \varphi(\llbracket \Phi \rrbracket(f)(x)) = \llbracket \Phi \rrbracket(f)(\varphi(x)) = f^*\varphi(x)$ . Hence  $\varphi = \llbracket \alpha \rrbracket$ .
- *essential surjectivity.* Let  $X \in \hat{C}_f$ . According to Lemma 1.28, X admits a finite number of generators  $x_i$   $(1 \le i \le n)$ . We proceed by induction on n.
  - Suppose n = 0, then X is the empty presheaf, and is the image of the empty context.
  - Suppose n > 0. Suppose  $x = x_n \in X_c$  is of maximal dimension. Let  $Y := X \setminus \{f^*x\}_{f \in C_-}$ . That is, Y is X without the degeneracies of x. Since C is eleguant, we may check that Y is again a presheaf. Moreover, the non-degenerated cells of Y are exactly the  $\{x_i\}_{1 \le i < n}$ , hence Theorem 1.29 shows that  $Y \in \hat{C}_f$ . There is an inclusion  $Y \hookrightarrow X$  and a map  $c \xrightarrow{x} X$  given by the YONEDA embedding. We see that those two maps make X the following pushout:



Let  $z : c \to Z$  (seen as  $z \in Z_c$ ) and  $\varphi : Y \to Z$  such that  $\varphi \circ x|_{\partial c} = z|_{\partial c}$ . If,  $\varphi$  and z factor through  $\psi : X \to Z$ , then  $\psi(x) = z$  and  $\psi|_Y = \varphi : Y \to Z$ , so  $\psi$  is completely defined as a natural transformation  $X \to Z$  by  $\psi(f^*x) = f^*z$  for  $f \in C_-$ .

We then check that  $\psi$  is natural. Let  $x' \in X$  and g such that  $g^*x'$  makes sense. Because  $\varphi$  is natural, if  $x' \in Y$ , we already have  $\psi(g^*x') = g^*\psi(x')$ . For x' = x, write  $g = g_+ \circ g_-$  for some  $g_+ \in C_+$  and  $g_- \in C_-$ . If  $g = g_-$ ,  $\psi(g^*x) = g^*\psi(x)$  is by definition. Assume  $g_+ \neq id$ .

$$\psi(g^*x') = \psi(g^*_-g^*_+x)$$
  
=  $g^*_-\psi(g^*_+x)$  because  $g^*_+x \in Y$   
=  $g^*_-g^*_+\psi(x)$  because  $\varphi \circ x|_{\partial c} = z|_{\partial c}$   
=  $g^*\psi(x)$ 

Then, if  $x' = f^*x$  for some  $f \in C_-$  and  $g \in C$ , the above property yield  $\psi(g^*x') = g^*f^*\psi(x') = g^*\psi(f^*x')$  whence the result.

Now, using the induction hypothesis with *Y* yields a context  $\Phi$  such that  $\llbracket \Phi \rrbracket \simeq Y$ , and Lemma 3.22 gives a type  $\tau$  in  $\Phi$  such that  $\llbracket \tau \rrbracket = x|_{\partial c}$ . Then Lemma 3.23 proves that  $X \simeq \llbracket \Phi, x : \tau \rrbracket$ , where  $\Phi, x : \tau$  is obtained by CTX-EXT from  $\Phi$ .

# **4** Set-valued models of **PRETTY**

In this section, our aim is to characterize the (set-valued) models of  $PRETTY_{C}$ . As expected, we will see that they corresponds precisely to presheaves over C. We start by noticing that the syntactic category of PRETTY

admits a structure of *category with families* (CwF). We refer the reader to [4] for a gentle introduction to this notion.

### Definition 4.1

From now on,  $S_{PRETTY}$  will be seen as the following CwF.

- *S*<sub>PRETTY</sub> is the underlying category.
- For a context  $\Phi$ ,  $\mathcal{T}y^{\Phi} = \{\tau \mid \Phi \vdash \tau \text{ type is derivable}\}.$
- For a context  $\Phi$  and  $\tau \in \mathcal{T}y^{\Phi}$ ,  $\mathcal{T}m_{\tau}^{\Phi} = \{t \mid \Phi \vdash t : \tau \text{ is derivable}\}.$
- The values of (*Ty*, *Tm*) on substitutions is given by the action of substitutions on types and terms.
- The specified terminal object of  $S_{PRETTY}$  is the empty context  $\emptyset$ .
- For a context  $\Phi$  and  $\tau \in \mathcal{T}_{\mathcal{V}}^{\Phi}$ , the context comprehension operation is given by:
  - The object ( $\Phi$ , *a* :  $\tau$ ) obtained by the rule ctx-EXT.
  - The substitution  $\pi = \langle x \mapsto x \rangle_{x \in \mathsf{FV}(\Phi)} : (\Phi, a : \tau) \to \Phi$ . where *a* is minimal in  $\mathcal{A} \setminus \mathsf{FV}(\Phi)$ .
  - The term  $a \in \mathcal{T}m_{\tau}^{(\Phi, a:\tau)}$  found as said above.

**4.2** For any object *c* of *C*, the presheaf hom(-, c) and representing *c* its boundary  $\partial c$  are finitely generated (*c.f.* Remark 1.14). Hence, using the equivalence  $S_{\text{PRETTY}} \simeq \hat{C}_f$  of Theorem 3.24, we let  $\Phi_c$  (resp.  $\Phi_{\partial c}$ ) denotes a context such that  $\llbracket \Phi_c \rrbracket \simeq \hom(-, c)$  (resp.  $\llbracket \Phi_{\partial c} \rrbracket \simeq \partial c$ ).

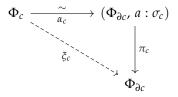
By definition of []], for any object *c* and any  $(u : b_u \rightarrow c) \in \mathcal{F}(c)$ , the judgment

$$\Phi_{\partial c} \vdash e(u) : (b_u, (e(v \circ u))_{v \in \mathcal{F}(b_u)})$$

is derivable, where  $e : \hom_{C_-}(-, c) \to A$  is an encoding of every  $f \in C_-(-, c)$  as a variable  $e(f) \in A$ . Then, using the rule TYPE, the judgment

$$\Phi_{\partial c} \vdash \sigma_c$$
 type

holds, where  $\sigma_c = (c, e(u)_{u \in \mathcal{F}(c)})$ . So  $\Phi_c$  is obtained by a context extension (using CTX-EXT) from  $\Phi_{\partial c}$ . In particular, there is an isomorphism  $\alpha_c$  and a display map  $\pi_c$  as follows:



such that  $\alpha_c$ ,  $\pi_c$  and  $\xi_c = \alpha_c \circ \pi_c$  are natural in *c*.

Lemma 4.3: Representability of types and terms

For any object  $c \in C$  and context  $\Psi$  of PRETTY, the map

$$\begin{array}{rcl} \mathcal{S}_{\mathrm{PRETTY}}(\Psi, \, \Phi_{\partial c}) & \to & \{\tau \in \mathcal{T}y^{\Psi} \mid \underline{\tau} = c\} \\ & \alpha & \mapsto & \sigma_c[\alpha] \end{array}$$

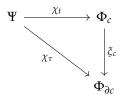
is an isomorphism, natural in  $\Psi$ . Given a type  $\tau$  with  $\underline{\tau} = c$ , we denote the associated substitution

$$\chi_{ au}: \Psi o \Phi_{\partial c}$$

We have moreover that the maps

$$\left( \mathcal{S}_{\text{PRETTY}} / \Phi_{\partial c} \right) \left( \chi_{\tau} : \Psi \to \Phi_{\partial c}, \, \xi_{c} : \Phi_{c} \to \Phi_{\partial c} \right) \quad \xrightarrow{} \quad \mathcal{T}m_{\tau}^{\Psi}$$
  
$$\alpha \quad \mapsto \quad e(\mathsf{id}_{c})[\alpha]$$

are also isomorphisms, natural in  $\Psi$ . Given a term  $t \in \mathcal{T}m_{\tau}^{\Psi}$ , we let  $\chi_t$  denotes the associated substitution over  $\Phi_{\partial c}$ , in such a way that the following triangle commutes.

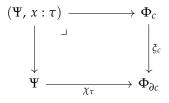


*Proof.* The first point is the reflection of Lemma 3.22 along the equivalence  $S_{PRETTY} \simeq \hat{C}_f^{op}$  of Theorem 3.24. The second is similarly the reflection of the point (i) of Lemma 3.18.

**4.4** Using Lemma 4.3, we may see the interpretation functor  $[\![-]\!] : S_{PRETTY} \to \hat{C}_f^{op}$  as follows. The functor  $\Phi_{\bullet} : C^{op} \to S_{PRETTY}$  allows us to consider a nerve functor  $N : S_{PRETTY} \to \hat{C}^{op}$  given by  $N(\Psi)_c = S_{PRETTY}(\Psi, \Phi_c)$ . Under the correspondance of Lemma 4.3, this functor coincides with  $[\![-]\!]$  on the contexts. Moreover, given a substitution  $\alpha : \Theta \to \Psi$ , and a term  $t \in [\![\Psi]\!]_c$ , the definition of  $\chi_t$  yields  $\chi_t \circ \alpha = \chi_t[\alpha]$ . On the other hand,  $[\![\alpha]\!](t) = t[\alpha]$ . So N and  $[\![-]\!]$  coincides modulo the natural equivalence of Lemma 4.3. Using this interpretation, the equivalence of Theorem 3.24 may be seen as coming from a generalised nerve - realisation adjunction.

#### Lemma 4.5

Any context  $(\Psi, x : \tau)$  where  $\underline{\tau} = c$  is obtained as a pullback, as follows:

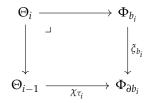


*Proof.* This is a direct consequence of Lemma 4.3 and the comprehension operation property.

#### Lemma 4.6

Let  $\mathcal{D}$  be a finitely complete category, and  $F : \mathcal{C}^{\text{op}} \to \mathcal{D}$  a functor. F extends uniquely to a functor  $\tilde{F} : S_{\text{PRETTY}} \to \mathcal{D}$  which preserves the terminal object and the pullbacks along display maps.

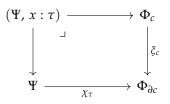
*Proof.* First of all we show that  $\tilde{F}$  is determined on the contexts  $\Phi_{\partial c}$  for all c, notice that in the context  $\Phi_{\partial c}$ , each pair  $(x : \tau)$  satisfies  $\underline{\tau} \triangleleft c$ . So we may show by induction that  $F(\Phi_{\partial c})$  is determined. Indeed, we may construct  $\Phi_{\partial c} = \Theta_n$  as a succession of pullbacks (using Lemma 4.5):



Where  $\Theta_k$  is the context of the first k elements of  $\Phi_{\partial c}$ , and  $(t_i : \tau_i)$  with  $\underline{\tau_i} = b_i$  is its k-th element. If we assume  $F(\Phi_{\partial b})$  to be known for each  $b \triangleleft c$ , then  $\Theta_n$  must be preserved as a tower of pullbacks along (maps isomorphic to) display maps. That is,  $F(\Phi_{\partial c})$  will be caracterised universally as a tower of pullbacks along the  $F(\Phi_{b_i}) \rightarrow F(\Phi_{\partial b_i})$ . In particular, it caracterise the image by F of morphisms whose target is  $\Phi_{\partial b_i}$ .

Now, we may prove by induction on the length of the context  $\Psi$  that  $F(\Psi)$  is also determined as a colimit in  $\mathcal{D}$ , and on the maps whose target is  $\Psi$ .

- On the empty context, this is true because *F* is assumed to send Ø to the empty set.
- Let  $(\Psi, x : \tau)$  be obtained by ctx-EXT. Then using Lemma 4.5, we have a pullback diagram in  $S_{PRETTY}$ :

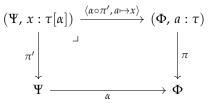


And since  $\xi_c$  is (isomorphic to) a display map, *F* must preserve this pullback. So  $F((\Psi, x : \tau))$  is defined as a pullback in  $\mathcal{D}$ , which ensure the desired property.

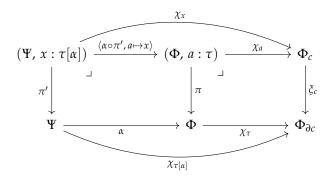
Now, we may check the functoriality of *F*. For every context  $\Theta$ ,  $F(\Theta)$  has been defined inductively as a pullback of some  $\xi_c$ . We will ensure the functoriality of *F* on maps whose target is  $\Theta$  by induction on this process.

- Suppose  $\Theta = \emptyset$ , then  $F(\emptyset) = \emptyset$  is terminal so the result is clear.
- Suppose that  $\Theta = (\Psi, x : \tau)$ , then it a pullback of  $\xi_c$  for  $c = \underline{\tau}$  along  $\chi_\tau$ , as depicted above. Our inductive hypothesis is that *F* is functorial on maps whose target are  $\Phi_c$  or  $\Psi$ . A map  $\alpha : \Theta' \to (\Psi, x : \tau)$  yields two structural maps  $\alpha_1 : \Theta' \to \Phi_c$  and  $\alpha_2 : \Theta' \to \Psi$ . By definition of *F*, *F*( $\alpha$ ) is obtained by universal property of *F*( $\Theta$ ), applied to the structural maps  $F(\alpha_1)$  and  $F(\alpha_2)$ . Let  $\beta : \Theta'' \to \Theta'$  be a substitution. Then similarly,  $\gamma = \alpha \circ \beta$  yield two maps  $\gamma_1 : \Theta'' \to \Phi_c$  and  $\gamma_2 : \Theta'' \to \Psi$ , and  $F(\gamma)$  is defined by the universal property of *F*( $\Theta$ ), applied to the maps *F*( $\gamma_1$ ) and *F*( $\gamma_2$ ). By induction hypothesis,  $F(\gamma_1) = F(\alpha_1) \circ F(\beta)$  and  $F(\gamma_2) = F(\alpha_2) \circ F(\beta)$ . Hence,  $F(\gamma) = F(\alpha) \circ F(\beta)$  by the universal property of *F*( $\Theta$ ). Hence, *F* preserves compositions. By definition of *F*, it also preserves identities.

Finally, we shall check that *F* preserves the terminal object and the pullbacks along display maps. The first point is by definition of *F*. As to the second one, consider a pullback along a display map in  $S_{PRETTY}$ , it has the following form:



Lemma 4.5 gives us two other pullback squares, as follows:



which are the rightmost and the outermost ones. Since both must be preserved by definition of F, the leftmost one must also be preserved, by the pullback pasting lemma.

#### Theorem 4.7

There is an equivalence of categories  $\mathbf{Mod}(\mathcal{S}_{\mathsf{PRETTY}}) \simeq \hat{\mathcal{C}}$ , by restriction along  $\mathcal{C}^{\mathsf{op}} \hookrightarrow \mathcal{S}_{\mathsf{PRETTY}}$ .

Proof. It is given directly by Lemma 4.6.

# **5** Some instances of **PRETTY**

### The Globular Theory PRETTYG

For this exemple, we consider the reflexive category of globes **G**. We have the coverings defined in Exemple 1.8, and use the presentation  $P([n + 1], [n]) = {\iota_n}$  With no further relations.

Note that we have  $\iota \circ \sigma = \iota \circ \tau = id$  when those expressions makes sense, so this choice of presentation satisfies the assumptions of 3.1.

**5.1 Syntax.** We may describe the syntax for terms and types as follows:

$$\begin{array}{rcl} \mathsf{tm} & \coloneqq & x & (x \in \mathcal{A}) \\ & \mid & \iota^* \mathsf{tm} \end{array}$$
$$\begin{array}{rcl} \mathsf{tp} & \coloneqq & * \\ & \mid & s \to t & (s,t \in \mathsf{tm}) \end{array}$$

We also write more conveniently  $\operatorname{id}_t^k$  for  $\iota^{*k}t$ , yielding  $\operatorname{tm} = {\operatorname{id}_x^k}_{x \in \mathcal{A}, k \in \mathbb{N}}$ 

5.2 Type introduction. The rule TYPE splits into the two following ones:

 $\frac{\Phi \operatorname{ctx}}{\Phi \vdash * \operatorname{type}} \xrightarrow{\operatorname{TYPE-*}} \frac{\Phi \operatorname{ctx}}{\Phi \vdash s : \tau} \xrightarrow{\Phi \vdash t : \tau} \xrightarrow{\operatorname{TYPE-*}} \xrightarrow{\operatorname{TYPE-*}}$ 

type introduction rules

5.3 Degeneracies. The rule DEGE boils down to:

$$\frac{\Phi \vdash t : \tau}{\Phi \vdash \mathsf{id}_t : t \to t} \text{ DEGE}$$

degeneracy rule

**5.4 Globular Type Theory.** Upon forgeting the degeneracies, one directly recovers the *Globular Type Theory* defined by BENJAMIN, FINSTER and MIMRAM in [2].

### The Simplicial Theory PRETTY<sub> $\Delta$ </sub>

We consider now the category of simplices  $\Delta$ . We have the coverings defined in Exemple 1.24, and use the presentation  $P([n + 1], [n]) = {\sigma_i}_{0 \le i \le n}$  With relations  $(\sigma_j \circ \sigma_i)P(\sigma_i \circ \sigma_{j+1})$  when  $i \le j$ . We have the identities

$$\sigma_j \circ \delta_i = \begin{cases} \delta_i \circ \sigma_{j-1} & \text{if } i < j \\ \text{id} & \text{if } i \in \{j, j+1\} \\ \delta_{i-1} \circ \sigma_j & \text{if } j < i+1 \end{cases}$$

when those expressions makes sense, so this choice of presentation satisfies the assumptions of 3.1. **5.5 Syntax.** We may describe the syntax for terms and types as follows:

$$\begin{array}{rcl} \mathsf{tm} & \mathop{:\!\!:}= & x & (x \in \mathcal{A}) \\ & \mid & \sigma_i^* \mathsf{tm} & (i \in \mathbb{N}) \end{array} \\ \\ \mathsf{tp} & \mathop{:\!\!:}= & * \\ & \mid & \Delta_{[s_0, \cdots, s_n]} & (n \ge 1, s_i \in \mathsf{tm}) \end{array}$$

We write more conveniently *t.i* for  $\sigma_i^* t$ , yielding tm =  $\{x.i_1......i_n\}_{x \in \mathcal{A}, n \in \mathbb{N}, i_k \in \mathbb{N}}$ . We also write  $s \to t$  for the type  $\Delta_{[s,t]}$ .

**5.6 Type introduction.** The rule TYPE splits into the three following ones:

$$\frac{\Phi \operatorname{ctx}}{\Phi \vdash * \operatorname{type}} \operatorname{TYPE-*} \qquad \frac{\Phi \operatorname{ctx} \quad \Phi \vdash s : \tau \quad \Phi \vdash t : \tau}{\Phi \vdash s \to t \operatorname{type}} \operatorname{TYPE-} \\ \frac{\bigwedge_{0 \leq i \leq n+1} \Phi \vdash t_i : \Delta_{[s_0^i, \cdots, s_n^i]}}{\Phi \vdash \Delta_{[t_0, \cdots, t_n]}} \qquad \bigwedge_{0 \leq i < j \leq n+1} s_i^j = s_i^j \\ \Phi \vdash \Delta_{[t_0, \cdots, t_n]} \operatorname{type} \operatorname{TYPE-} \Delta \quad (n \geq 2)$$

$$type introduction rules$$

**5.7 Degeneracies.** The rule DEGE splits into the following ones:

$$\frac{\Phi \vdash t: *}{\Phi \vdash t.0: t \to t} \text{ DEGE-*} \qquad \frac{\Phi \vdash t: \Delta_{[s_0, \cdots, s_n]}}{\Phi \vdash t.k: \Delta_{[s_0.(k-1), \cdots, s_{k-1}.(k-1), t, t, s_{k+1}.k, \cdots, s_n.k]}} \text{ DEGE-} \Delta \quad (0 \le k \le n)$$

degeneracy rules

The equality  $s \equiv t$  of two terms may be decided quickly by using the following argument:

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Lemma 5.8: Normal form of terms
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Each term  $t \in \text{tm}$  may be written as  $x.i_1.....i_n$  for a unique x, n and  $i_k$ 's such that  $i_1 \leq \cdots \leq i_n$ .

# The Cubical Theory PRETTY□

We finally consider the exemple of the category of cubes  $\Box$ . We have the coverings defined in Exemple 1.25, and use the presentation  $P([n + 1], [n]) = \{\sigma_i\}_{0 \le i \le n}$  With relations  $(\sigma_j \circ \sigma_i)P(\sigma_i \circ \sigma_{j+1})$  when  $i \le j$ . We have the identities

$$\sigma_{j} \circ \delta_{i,\varepsilon} = \begin{cases} \delta_{i,\varepsilon} \circ \sigma_{j-1} & \text{if } i < j \\ \text{id} & \text{if } i = j \\ \delta_{i-1,\varepsilon} \circ \sigma_{j} & \text{if } i > j \end{cases}$$

when those expressions makes sense, so this choice of presentation satisfies the assumptions of 3.1. **5.9 Syntax.** We may describe the syntax for terms and types as follows:

$$\begin{array}{rcl} \operatorname{tm} & ::= & x & (x \in \mathcal{A}) \\ & \mid & \sigma_i^* \operatorname{tm} & (i \in \mathbb{N}) \end{array} \end{array}$$
$$\operatorname{tp} & ::= & \Box_{[(s_1^0, s_1^1), , \cdots, (s_n^0, s_n^1)]} & (n \ge 0, \, s_i^{\varepsilon} \in \operatorname{tm}) \end{array}$$

We write more conveniently *t.i* for  $\sigma_i^* t$ , yielding tm =  $\{x.i_1......i_n\}_{x \in \mathcal{A}, n \in \mathbb{N}, i_k \in \mathbb{N}}$ . We also write \* for the type  $\Box_{||}$ .

**5.10 Type introduction.** The rule TYPE splits into the two following ones:

$$\begin{split} & \bigwedge_{\substack{1 \leq i \leq n \\ 0 \leq \varepsilon \leq 1}} \Phi \vdash t_i^{\varepsilon} : \Box_{[(s_1^{\varepsilon,i,0}, s_1^{\varepsilon,i,1}), \cdots, (s_n^{\varepsilon,i,0}, s_n^{\varepsilon,i,1})]} & \bigwedge_{\substack{1 \leq i < j \leq n \\ 0 \leq \varepsilon, \eta \leq 1}} s_{j-1}^{\varepsilon,i,\eta} = s_i^{\eta,j,\varepsilon} \\ & \xrightarrow{0 \leq \varepsilon, \eta \leq 1} & \text{TYPE-} \Box \quad (n \geq 0) \\ \hline \Phi \vdash \Box_{[(s_1^0, s_1^1), \cdots, (t_n^0, t_n^1)]} \text{ type} & \text{TYPE-} \Box \quad (n \geq 0) \end{split}$$

**5.11 Degeneracies.** The rule DEGE may be written as follows:

 $\Phi dash t: \Box_{[(s_1^0, s_1^1), \cdots, (s_n^0, s_n^1)]}$ 

- DEGE  $(1 \le k \le n)$ 

 $\overline{\Phi \vdash t.k: \Box_{[(s_1^0.(k-1), s_1^1.(k-1)), \cdots, (s_{k-1}^0.(k-1), s_{k-1}^1.(k-1)), (t,t), (s_{k+1}^0.k, s_{k+1}^1.k), \cdots, (s_n^0, s_n^1).k]}} \ \mathsf{L}_{k+1}$ 

degeneracy rules

The equality  $s \equiv t$  of two terms may be decided quickly by using the following argument:

Lemma 5.12: Normal form of terms

Each term  $t \in \text{tm}$  may be written as  $x.i_1.....i_n$  for a unique x, n and  $i_k$ 's such that  $i_1 \leq \cdots \leq i_n$ .

# References

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