A Type Theory for Presheaves Over a Reedy Category

Louise Leclerc

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Abstract

We introduce a small type theory whose models are precisely the pesheaves over a given Reedy category $\mathcal{C}$ with a given system of *coverings*, satisfying a certain assumption of local finiteness and presentability. Our work is directly inspired from the *Globular Type Theory* of Benjamin, Finster and Mimram [2], and the *Simplicial Type Theory* of Riehl and Shulman [5].

1 Some category-theoretic definitions and results

Direct categories

We first recall the notion of direct category, then we introduce some tools that we will need later on.

Definition 1.1 : direct category

A category $\mathcal{D}$ is said to be a *direct category* when the following order $\lhd$ on $\mathcal{O}b(\mathcal{D})$ is well-founded.

$$a \lhd b \iff (\exists f : a \to b,\ f \neq \mathsf{id}_a)$$

Remark 1.2
Note that a direct category has neither non-identity isomorphisms nor non-identity endomorphisms.

1.3 coverings. Let $\mathcal{C}$ be a direct category with finite slices $\mathcal{C}/c$ for $c \in \mathcal{O}b(\mathcal{C})$ (that is a *locally finite* direct category), then we let for $c \in \mathcal{O}b(\mathcal{C})$ the *covering* $\mathcal{F}(c)$ of c be defined as the set of non-identity morphisms $u : b \to c$ such that u has no non-trivial factorization.

$$\mathcal{F}(c) := \{u : b \to c \mid f \neq \mathsf{id} \land \neg(\exists (g, v),\ g \neq \mathsf{id},\ v \neq \mathsf{id},\ u = v \circ g)\}$$

We write such a morphism $(u : b \to c) \in \mathcal{F}(c)$ as $u : b \rightarrowtail c$. We write $\mathcal{F}^*(c)$ for $\mathcal{F}(c) \cup \{\mathsf{id}_c\}$.

Since $\mathcal{C}$ is assumed to have finite slices, we may observe that each non-identity morphism $f : a \to c$ factors as

$$a \xrightarrow{\quad g \quad} b \overset{v}{\rightarrowtail} c$$

And each $u \in \mathcal{F}(c)$ admits a unique such factorization, where $v = u$ and $g = \mathsf{id}$.

Definition 1.4 : Monic direct category

A direct category $\mathcal{C}$ is said to be *monic* if all its morphisms are monomorphisms.

Remark 1.5
Notice that when $\mathcal{C}$ is locally finite, it equivalent to ask that the morphisms of $\mathcal{F}$ are monomorphisms.

Definition 1.6 : two-layered boundaries

See Definition 1.19 for the more general case.

1.7 We are now going to present three important exemples, note that you may find some further references on those on the nlab (see [1])

Exemple 1.8 (The category $\mathbf{G}_+$ of globes)
We consider the following category $\mathbf{G}_+$.

$$\begin{tikzcd} {[0]} \arrow[r, shift left, "\sigma_0"] \arrow[r, shift right, "\tau_0"'] & {[1]} \arrow[r, shift left, "\sigma_1"] \arrow[r, shift right, "\tau_1"'] & {[2]} \arrow[r, shift left, "\sigma_2"] \arrow[r, shift right, "\tau_2"'] & {[3]} \arrow[r, shift left, "\sigma_3"] \arrow[r, shift right, "\tau_3"'] & \cdots \end{tikzcd}$$

whose set of objects is isomorphic to $\mathbb{N}$ (we denote $[i]$ the i-th object), generated by the morphisms σ_i, $\tau_i : [i] \to [i+1]$ subject to the *coglobular relations*:

$$(i \in \mathbb{N}) \qquad \sigma_{i+1} \circ \sigma_i = \tau_{i+1} \circ \sigma_i \qquad \sigma_{i+1} \circ \tau_i = \tau_{i+1} \circ \tau_i$$

The maps σ_i ($i \in \mathbb{N}$) are called the *cosources* and the τ_i ($i \in \mathbb{N}$) the *cotargets*.

The order $\lhd$ of Definition 1.1 is isomorphic to ω, so $\mathbf{G}_+$ is a direct category. It is locally finite and monic, its coverings are given by $\mathcal{F}([0]) = \varnothing$ and $\mathcal{F}([n+1]) = \{\sigma_n, \tau_n\}$. And it has two-layered boundaries, indeed, the pullback of σ_0, $\tau_0 : [0] \rightrightarrows [1]$ is the empty sum, and the pullback of σ_{n+1}, $\tau_{n+1} : [n+1] \rightrightarrows [n+2]$ is given by $[n] + [n]$.

$$\begin{tikzcd} {[n+1]} \arrow[r, tail, "\sigma_{n+1}"] & {[n+2]} \\ {[n]+[n]} \arrow[u, "{[\sigma_n, \tau_n]}"] \arrow[r, "{[\sigma_n, \tau_n]}"'] \arrow[ur, phantom, very near start, "\urcorner"] & {[n+1]} \arrow[u, tail, "\tau_{n+1}"'] \end{tikzcd}$$

Exemple 1.9 (The category $\mathbf{\Delta}_+$ of simplices)
We consider the category $\mathbf{\Delta}_+$ whose objects are the non-empty finite ordered sets $[i] = \{0 < 1 < \cdots < i\}$ and whose maps are the increasing maps, called the *cofaces*. As for the category $\mathbf{G}$, the order $\lhd$ of Definition 1.1 is isomorphic to ω, so $\mathbf{\Delta}_+$ is a direct category. It is locally finite and monic, its coverings are given by $\mathcal{F}([0]) = \varnothing$ and $\mathcal{F}([n+1]) = \{\delta_i\}_{0 \le i \le n+1}$ where d_i is the only increasing map $[n] \to [n+1]$ such that $i \notin \delta_i([n])$. And it has two-layered boundaries, indeed, the pullback of δ_0, $\delta_1 : [0] \rightrightarrows [1]$ is the empty sum, and the pullback of δ_i, $\delta_j : [n+1] \rightrightarrows [n+2]$ ($i < j$) is given by $[n]$.

$$\begin{tikzcd} {[n+1]} \arrow[r, tail, "\delta_i"] & {[n+2]} \\ {[n]} \arrow[u, tail, "\delta_{j-1}"] \arrow[r, tail, "\delta_i"'] \arrow[ur, phantom, very near start, "\urcorner"] & {[n+1]} \arrow[u, tail, "\delta_j"'] \end{tikzcd}$$

Exemple 1.10 (The category $\square_+$ of cubes)
We consider the category $\square_+$ whose objects are the sets $[i] = \{0, 1\}^i$ for $i \ge 0$. And whose maps are the functions $[i] \to [j]$ which inserts 0 or 1 along a tuple. That is, maps

$$\delta_{k_1, \varepsilon_1, \cdots, k_{j-i}, \varepsilon_{j-i}} : [i] \longrightarrow [j]$$

described as

$$[i] \simeq [1]^i \to [1]^{k_1 - 1} \times \{\varepsilon_1\} \times [1]^{k_2 - k_1 - 1} \times \cdots \times \{\varepsilon_{j-i}\} \times [1]^{j - k_{j-i}} \hookrightarrow [1]^j \simeq [j] \quad .$$

Once again, the order $\lhd$ is isomorphic to ω, making $\square_+$ a direct category. It is also locally finite and monic, its coverings are given by $\mathcal{F}([0]) = \varnothing$ and $\mathcal{F}([n+1]) = \{\delta_{i,\varepsilon}\}_{0 \le i \le n+1, 0 \le \varepsilon \le 1}$. Where

$$\begin{array}{rrcl} \delta_{i,\varepsilon} : & [n] & \longrightarrow & [n+1] \\ & (x_1, \cdots, x_n) & \longmapsto & (x_1, \cdots, x_{i-1}, \varepsilon, x_i, \cdots, x_n) \end{array}$$

For two maps $\delta_{i,0} : [n] \to [n+1]$ and $\delta_{i,1} : [n] \to [n+1]$, their pullback in $\hat{\square}_+$ is the empty presheaf. And for any two maps $\delta_{i,\varepsilon} : [n+1] \to [n+2]$ and $\delta_{j,\eta} : [n+1] \to [n+2]$ where $i < j$, their pullback is given as follows:

$$\begin{tikzcd} {[n+1]} \arrow[r, tail, "\delta_{i,\varepsilon}"] & {[n+2]} \\ {[n]} \arrow[u, tail, "\delta_{j-1,\eta}"] \arrow[r, tail, "\delta_{i,\varepsilon}"'] \arrow[ur, phantom, very near start, "\urcorner"] & {[n+1]} \arrow[u, tail, "\delta_{j,\eta}"'] \end{tikzcd}$$

Hence $\square_+$ has two-layered boundaries.

Definition 1.11 : Boundaries

Let $\mathcal{C}$ be a direct category and $c \in \mathcal{O}b(\mathcal{C})$ an object. The *boundary* of c, denoted ∂c, is the presheaf on $\mathcal{C}$ defined by $\partial c(b) = \{f : b \to c \mid f \neq \mathsf{id}\}$.

Proposition 1.12 : Decomposition of boundaries

See Proposition 1.18 for the more general case.

Definition 1.13 : $\hat{\mathcal{C}}_f$

Let $\mathcal{C}$ be a category. We let $\hat{\mathcal{C}}_f$ denotes the category of finite colimits of representable presheaves over the category $\mathcal{C}$. We call $\hat{\mathcal{C}}_f$ the category of *finitely generated* presheaves.

Remark 1.14
Note that $\hat{\mathcal{C}}_f$ is a full subcategory of $\hat{\mathcal{C}}$ which contains the boundaries. When $\mathcal{C}$ is a locally finite and direct category, since any representable presheaf is finite and any finite presheaf may be expressed as a finite colimit of representable presheaves, $\hat{\mathcal{C}}_f$ may alternatively be described as the category of finite presheaves over $\mathcal{C}$.

REEDY categories

We now get to the more general case of REEDY categories, introduce the notion of *eleguant* REEDY category and study some of their properties.

Definition 1.15 : REEDY category

A REEDY category $\mathcal{C}$ consist of the following data.

- A category $\mathcal{C}$
- Two *wide* subcategories $\mathcal{C}_+$ and $\mathcal{C}_-$, where *wide* means "with the same objects than $\mathcal{C}$".
- A *degree* function $\mathsf{deg} : \mathcal{O}b(\mathcal{C}) \to \alpha$ for some ordinal α.

Such that

- Every morphism of $\mathcal{C}_-$ (resp. $\mathcal{C}_+$) lowers (resp. increases) the degree.
- $(\mathcal{M}or(\mathcal{C}_-), \mathcal{M}or(\mathcal{C}_+))$ is a *strict factorization system*, that is: every map f factors uniquely as $f_+ \circ f_-$ where $f_- \in \mathcal{M}or(\mathcal{C}_-)$ and $f_+ \in \mathcal{M}or(\mathcal{C}_+)$.

Remark 1.16
From those properties, one see that $\mathcal{C}_+ \cap \mathcal{C}_-$ contains exactly the identities. Moreover, the category $\mathcal{C}_+$ is always a direct category.

Definition 1.17 : Boundaries

Let $\mathcal{C}$ be a REEDY category and $c \in \mathcal{O}b(\mathcal{C})$ an object. The *boundary* of c, denoted ∂c, is the presheaf on $\mathcal{C}$ defined by $\partial c(b) = \{f : b \to c \mid f \text{ factors through } \mathcal{F}(c)\}$.

Proposition 1.18 : Decomposition of boundaries

Let $c \in \mathcal{O}b(\mathcal{C})$ an object of a Reedy category, and let $(a(u, v), p_1, p_2)$ be the choice of a pullback in $\hat{\mathcal{C}}$ for any two maps $u \neq v \in \mathcal{F}(c)$. Then ∂c may be seen as the colimit

$$\partial c = \operatorname{colim}\left(\mathcal{D}_c \to \hat{\mathcal{C}}\right)$$

Where $\mathcal{D}_c$ is the category whose objetcs are

$$\mathcal{O}b(\mathcal{D}_c) = \{b_u\}_{u \in \mathcal{F}(c)} \cup \{a(u, v)\}_{u \neq v \in \mathcal{F}(c)}$$

and the arrows are the legs p_1, p_2 of each pullback. The structural maps of the colimit being given by the $u : b_u \to \partial c$ obtained by the Yoneda lemma.

Definition 1.19 : two-layered boundaries

A locally finite Reedy category $\mathcal{C}$ will be said to have *two-layered boundaries* if for any two distinct $u_1, u_2 \in \mathcal{F}(c)$, the pullback of $u_1 : b_1 \to c$ and $u_2 : b_2 \to c$ (well defined in $\hat{\mathcal{C}}$) decomposes as a (finite) coproduct of representable presheaves, and the projections p_1, p_2 of this pullback decompose accordingly as morphisms in $\mathcal{F}$.
That is, there is a finite set $\mathcal{J}_{u_1,u_2}$ such that the coproduct $a(u_1, u_2)$ of the $a_j(u_1, u_2)$ ($j \in \mathcal{J}_{u_1,u_2}$) in $\hat{\mathcal{C}}$ is the aforementioned pullback. Then, writing $p_{i,j}(u_1, u_2)$ for the j-th component of the projection $p_i(u_1, u_2) : a(u_1, u_2) \to b_i$, we have $p_{i,j}(u_1, u_2) \in \mathcal{F}(b_i)$ for every i, j.

$$\begin{tikzcd}
b_1 \arrow[r, tail, "u_1"] & c \\
a(u_1, u_2) \arrow[u, "p_1(u_1,u_2)"] \arrow[r, "p_2(u_1,u_2)"'] \arrow[ur, phantom, "\urcorner"] & b_2 \arrow[u, tail, "u_2"']
\end{tikzcd}$$

Remark 1.20
Notice that such a pullback is always a finite coproduct of representable presheaves, because it may be expressed as a weighted colimit of such, with a weight whose category of element is finite. Moreover, if it decomposes as a coproduct of representable presheaves, then this decomposition is unique, hence canonical. So having two-layered boundaries is a property, and not an additional structure on a Reedy category $\mathcal{C}$.

Definition 1.21 : Degenerated cell

Let $X \in \hat{\mathcal{C}}$ where $\mathcal{C}$ is a Reedy category, we say that a *cell* x of X (that is an element $x \in X(c)$ for some $c \in \mathcal{O}b(\mathcal{C})$) is *degenerated* iff it may be written p^*y for some other cell y and $p \in \mathcal{C}_-$.
A cell is said to be *non-degenerated* iff it is not degenerated.

Definition 1.22 : Eleguant Reedy category

A Reedy category $\mathcal{C}$ is said to be *eleguant* if every cell of any presheaf $X \in \hat{\mathcal{C}}$ may be written as p^*x for a unique pair (p, x) of a map $p \in \mathcal{M}or(\mathcal{C}_-)$ and a cell x which is non-degenerated.

Exemple 1.23 (The category G of reflexive globes)
The category $\mathbf{G}$ has the same objects as $\mathbf{G}_+$ (Exemple 1.8). The degree function is given by $\deg([n]) = n$, $\mathbf{G}_+$ is the category of Exemple 1.8, and $\mathbf{G}_-$ is generated by the maps $\iota : [n+1] \to [n]$ subject to the relations $\iota \circ \sigma = \mathrm{id}$ and $\iota \circ \tau = \mathrm{id}$. One may check that $\mathcal{G}$ is an eleguant Reedy category with two-layered boundaries (given as for $\mathcal{G}_+$).

Exemple 1.24 (The category Δ of simplices with degeneracies)
The category $\mathbf{\Delta}$ has the same objects as $\mathbf{ffi}_+$ (Exemple 1.9). The degree function is given by $\deg([n]) = n$, $\mathbf{\Delta}_+$

is the category of Exemple 1.9, and $\mathbf{\Delta}_-$ is generated by the maps $s_i : [n+1] \to [n]$ $(0 \leq i \leq n)$ subject to the relations

- $\sigma_j \circ \sigma_i = \sigma_i \circ \sigma_{j+1}$ when $i \leq j$
- $\sigma_j \circ \delta_i = \begin{cases} \delta_i \circ \sigma_{j-1} & \text{if } i < j. \\ \mathsf{id} & \text{if } i \in \{j, j+1\}. \\ \delta_{i-1} \circ \sigma_j & \text{if } i > j+1. \end{cases}$

One may check that $\mathbf{\Delta}$ is an eleguant REEDY category with two-layered boundaries (given as for $\mathbf{\Delta}_+$).

Exemple 1.25 (The category $\square$ of cubes with degeneracies)
The category $\square$ has the same objects as $\square_+$ (Exemple 1.10). The degree function is given by $\deg([n]) = n$, $\square_+$ is the category of Exemple 1.10, and $\square_-$ is generated by the maps $s_i : [n+1] \to [n]$ $(0 \leq i \leq n)$ subject to the relations

- $\sigma_j \circ \sigma_i = \sigma_i \circ \sigma_{j+1}$ when $i \leq j$
- $\sigma_j \circ \delta_i = \sigma_j \circ \delta_{i,\varepsilon} = \begin{cases} \delta_{i,\varepsilon} \circ \sigma_{j-1} & \text{if } i < j \\ \mathsf{id} & \text{if } i = j \\ \delta_{i-1,\varepsilon} \circ \sigma_j & \text{if } i > j \end{cases}$

(Please refer to [3] for a more precise presentation of the category of cubes.) One may check that $\square$ is an eleguant REEDY category with two-layered boundaries (given as for $\square_+$).

1.26 In the following, we fix a locally finite and eleguant REEDY category $\mathcal{C}$ indexed over ω. We will characterize the category $\hat{\mathcal{C}}_f$ of its finitely generated presheaves.

Lemma 1.27

If X is a representable presheaf over $\mathcal{C}$, then it has a finite number of non-degenerated cells.

Proof. Suppose $X = \mathsf{hom}(-, c)$. Then a cell $x : b \to c$ is non-degenerated iff it does not factor as $y \circ p$ for some $p : b \to a$ in $\mathcal{C}_-$ a non-identity. By the factorization property of REEDY categories, this happens iff $x \in \mathcal{C}_+$. However, because $\mathcal{C}_+$ is direct and locally finite, the set $\bigsqcup_{b \in \mathcal{O}b(\mathcal{C})}(\mathsf{hom}_{\mathcal{C}_+}(b, c))$ is finite. Whence the result. □

Lemma 1.28

Let $X \in \hat{\mathcal{C}}_f$, then X admits a finite set of non-degenerated elements $\{x_i\}_{1 \leq i \leq n}$. Moreover, each element $y \in X$ is a unique degeneracy of a unique x_i. We call the x_i's the *generators* of X.

Proof. Assuming the first statement to be proven, the second one is by definition of an eleguant Reedy category. As to the first one, let $F : \mathcal{I} \to \mathcal{C}$ be a finite diagram in $\mathcal{C}$, seen as a diagram in $\hat{\mathcal{C}}$. Suppose that X is a colimit of F, then for all $c \in \mathcal{O}b(\mathcal{C})$, X_c may be expressed as the quotient $\bigsqcup_{i \in \mathcal{O}b(\mathcal{I})} F(i)_c/_\sim$ where $\sim$ is the identification $x \sim F(\alpha)(x)$ for every $\alpha \in \mathcal{M}or(\mathcal{I})$.
Notice that the set X_{nd} of non-degenerated cells in X is included in the set $[\bigsqcup_{i \in \mathcal{O}b(\mathcal{I})} F(i)_{\mathsf{nd}}]$ of classes of non-degenerated cells of the $F(i)'s$.
We then conclude that it is finite using Lemma 1.27 and the finiteness of $\mathcal{I}$. □

Theorem 1.29

A presheaf $X \in \hat{\mathcal{C}}$ is finitely generated iff its set of non-degenerated cells is finite.

Proof. Lemma 1.28 gives the first implication. We now see the converse one.
Suppose that X has a finite set of non-degenerated cells $X_{\mathsf{nd}} = \{x_i\}_{1 \leq i \leq n}$. Let $\mathcal{P}_X$ denotes the full subcategory of $\int X$ whose objects are the $f^* x_i$ for some i and $f \in \mathcal{C}_+$. For $h \in \mathcal{M}or(\mathcal{C})$, there is a morphism $h : f^* x_i \to g^* x_j$ in $\mathcal{P}_X$ iff $h^* g^* x_j = f^* x_i$. Our assumptions on $\mathcal{C}$ ensures that $\mathcal{P}_X$ is a finite category. Let $\pi : \mathcal{P} \to \mathcal{C}$ be the canonical projection, sending $f^* x_i \in X_b$ to b, and a morphism h to itself. Then the YONEDA

embedding $Y : \mathcal{C} \to \hat{\mathcal{C}}$ gives a cocone $\pi \Rightarrow \Delta_X$, where Δ_X denotes the constant functor of value X. We now check that this cocone is universal.
Let $\varphi : \pi \Rightarrow Y$ for some $Y \in \hat{\mathcal{C}}$, if φ factors through $\psi : X \to Y$, then $\psi(x_i) = \varphi(x_i)$ for all i. Using the YONEDA embedding, we may see $\varphi(z)$ as an element in Y_b when $z \in X_b$. Hence, ψ is entirely and well-defined as a function, by $\psi(f^*x_i) = f^*\varphi(x_i)$ for any i and $f \in \mathcal{C}_-$.
Now, we check the naturality of ψ. First consider some $x_i \in X_{\mathsf{nd}}$ and some $f \in \mathcal{M}or(\mathcal{C})$ such that f^*x_i is well defined. We write $f = f_+ \circ f_-$ for $f_+ \in \mathcal{C}_+$ and $f_- \in \mathcal{C}_-$, then we check $\psi(f^*x_i) = f^*\psi(x_i)$. The cell $f_+^*x_i$ may be written p^*x_j for some j and $p \in \mathcal{C}_-$. By definition of $\mathcal{P}_X$, there is an arrow $p : g_+^*x_i \to x_j$ in $\mathcal{P}_X$. Hence, $\varphi(f_+^*x_i) = p^*\varphi(x_j)$. Moreover, $f^*x_i = f_-^*p^*x_j$, then

$$\psi(f^*x_i) = \psi(f_-^*p^*x_j) = f_-^*p^*\varphi(x_j) = f_-^*\varphi(f_+^*x_i) = f_-^*f_+^*\varphi(x_i) = f^*\psi(x_i)$$

Let z be any cell of X, written $z = f^*x_i$ for some i and $f \in \mathcal{C}_-$. Let $g \in \mathcal{M}or(\mathcal{C})$ such that g^*z makes sense. Then we shall check that $\psi(g^*z) = g^*\psi(z)$. That is, $\psi(g^*f^*x_i) = g^*\psi(f^*x_i)$. By the previous computation, $\psi(f^*x_i) = f^*\psi(x_i)$ and $\psi(g^*f^*x_i) = g^*f^*\psi(x_i)$ holds, whence the result. □

2 Presheaves over direct categories

In this section we fix a locally finite, monic direct category $\mathcal{C}$ with two-layered boundaries. And we will define a type theory whose contexts are the finite presheaves on $\mathcal{C}$. Its models will corresponds to the presheaves $\hat{\mathcal{C}}$. In the following, we let $\mathcal{J}_{u,v}$ and $p_{i,j}(u, v)$ be the indexing sets and legs for the chosen pullbacks of two maps u, v of $\mathcal{F}$, sticking with the notations of Definition 1.19.

The type theory

We first define the formal system, which will constitute the type theory. We will refer to it as PRETTY$_+$, a short for *Presheaf Type Theory*, where the $+$ refers to the case of direct categories. If we need to precise the direct category $\mathcal{C}$ we are working with, we will write PRETTY$_\mathcal{C}$.

2.1 Syntax. We begin by defining the notions of *terms*, *types*, *contexts* and *substitutions* for our type theory. First, we assume having an infinite and well ordered set of *variables*, which we write $(\mathcal{A}, \leq)$. We may take $\mathcal{A} = \omega$. When considering an implementation of the theory, we shall ask for a decidable equality on $\mathcal{A}$.

- A *term* (denoted t, $s \ldots$) is an element of $\mathcal{A}$ (that is a variable).
- A *type* (denoted τ, $\sigma \ldots$) is a pair $\tau = (c, (t_u)_{u \in \mathcal{F}(c)})$ where $c \in \mathcal{O}b(\mathcal{C})$ and t_u are terms. We let $\underline{\tau} = c$ and $\tau(u) = t_u$ for $u \in \mathcal{F}(c)$.
- A *context* (denoted Φ, $\Psi \ldots$) is a list $(x_1{:}\tau_1, \cdots, x_n{:}\tau_n)$ where the x_i's are variables and the τ_i's are types. The empty context is denoted $\varnothing$.
- A *substitution* (denoted α, $\beta \ldots$) is a list $\langle x_1 \mapsto t_1, \cdots, x_n \mapsto t_n \rangle$ where the x_i's are variables and the t_i's are terms. The empty substitution is denoted $\langle\rangle$.

2.2 Judgments. There are several kinds of derivable judgment in our theory. Informally:

- The judgment "Φ ctx" expresses that Φ is a well-formed context.
- The judgment "$\Phi \vdash \tau$ type" expresses that the type τ is a well-formed type in the context Φ.
- The judgment "$\Phi \vdash t : \tau$" expresses that t is a term of type τ in the context Φ.
- The judgment "$\Phi \vdash \alpha : \Psi$" expresses that α is a substitution of type Ψ in the context Φ (we also say that α is a substitution *from* Φ *to* Ψ).

2.3 Free variables. We define by induction on the syntax the set $\mathsf{FV}(x) \subseteq \mathcal{A}$ of *free variables* for x a term, type, context or substitution.

- *on terms (and variables):* $\mathsf{FV}(t) = \{t\}$.
- *on types:* $\mathsf{FV}((c, (t_u)_{u \in \mathcal{F}(c)})) = \bigcup_{u \in \mathcal{F}(c)} \mathsf{FV}(t_u)$.

- *on contexts:* $\mathsf{FV}((x_1{:}\tau_1, \cdots, x_n{:}\tau_n)) = \{x_i\}_{1\leq i\leq n}$.
- *on substitutions:* $\mathsf{FV}(\langle x_1 \mapsto t_1, \cdots, x_n \mapsto t_n\rangle) = \bigcup_{1\leq i\leq n} \mathsf{FV}(t_i)$.

2.4 Substitutions in terms and types. We define, for a term t (resp. a type τ), the *action* of a substitution $\alpha = \langle x_1 \mapsto t_1, \cdots, x_n \mapsto t_n\rangle$ on the term t (resp. the type τ), denoted $t[\alpha]$ (resp. $\tau[\alpha]$).

- *for t a term:* $t[\alpha] = \begin{cases} t_i & \text{if } t = x_i \text{ for some } i. \\ t & \text{in the other cases.} \end{cases}$
- *for $\tau = (c, (t_u))$ a type:* $\tau[\alpha] = (c, (t_u[\alpha]))$.

2.5 Identities and compositions for substitutions. For any context $\Phi = (x_1{:}\tau_1, \cdots, x_n{:}\tau_n)$, we may define a substitution, called the *identity* substitution on Φ as

$$\mathsf{id}_\Phi = \langle x_1 \mapsto x_1, \cdots, x_n \mapsto x_n\rangle.$$

and given two substitutions α and $\beta = \langle x_1 \mapsto t_1, \cdots, x_n \mapsto t_n\rangle$, we may define their composition $\beta \circ \alpha$ as

$$\langle x_1 \mapsto t_1[\alpha], \cdots, x_n \mapsto t_n[\alpha]\rangle$$

2.6 Inference rules. We give below the inference rules for the type theory:

$$\frac{}{\varnothing\ \mathsf{ctx}}\ \text{CTX-EMP} \qquad \frac{\Phi \vdash \tau\ \mathsf{type}}{(\Phi, x{:}\tau)\ \mathsf{ctx}}\ \text{CTX-EXT} \quad (x \notin \mathsf{FV}(\Phi))$$

context rules

$$\frac{(\Phi, x{:}\tau)\ \mathsf{ctx}}{\Phi, x{:}\tau \vdash x : \tau}\ \text{VAR} \qquad \frac{\Phi \vdash \tau_1\ \mathsf{type} \qquad \Phi \vdash t : \tau_2}{\Phi, x{:}\tau_1 \vdash t : \tau_2}\ \text{WKG} \quad (x \notin \mathsf{FV}(t) \cup \mathsf{FV}(\Phi))$$

typing rules

$$\frac{\Phi\ \mathsf{ctx} \qquad \bigwedge_{u \in \mathcal{F}(c)} \Phi \vdash t_u : \tau_u \qquad \bigwedge_{\substack{u_1 \neq u_2 \in \mathcal{F}(c) \\ j \in \mathcal{J}_{u_1,u_2}}} \tau_{u_1}(p_{1,j}) = \tau_{u_2}(p_{2,j})}{\Phi \vdash (c, (t_u)_{u \in \mathcal{F}(c)})\ \mathsf{type}}\ \text{TYPE}$$

type introduction rule

$$\frac{\Phi\ \mathsf{ctx}}{\Phi \vdash \langle\rangle : \varnothing}\ \text{SUB-EMP} \qquad \frac{\Phi \vdash \alpha : \Psi \qquad (\Psi, x{:}\tau)\ \mathsf{ctx} \qquad \Phi \vdash t : \tau[\alpha]}{\Phi \vdash \langle \alpha, x \mapsto t\rangle : (\Psi, x{:}\tau)}\ \text{SUB-EXT}$$

substitution rules

Some properties of PreTTy$_+$

We now expose some properties of the theory and study its syntactic category. Since PreTTy$_+$ is a special case of PreTTy which we will define later on, we postpone most of the proofs to the more general setting.

Lemma 2.7

The following properties may be shown by induction on the derivation trees.

- If "$\Phi \vdash \tau$ type" is derivable, so is "Φ ctx" and $\mathsf{FV}(\tau) \subseteq \mathsf{FV}(\Phi)$.
- If "$\Phi \vdash t : \tau$" is derivable, so is "$\Phi \vdash \tau$ type" and $\mathsf{FV}(t) \subseteq \mathsf{FV}(\Phi)$.

- If "$\Phi \vdash \alpha : \Psi$" is derivable, so are "$\Phi$ ctx" and "Ψ ctx".
- If "$\Phi \vdash (c, (t_u)_{u \in \mathcal{F}(c)})$ type" is derivable, all terms t_u are typeable is the context Φ.

Lemma 2.8 : Uniqueness of type

In a given context Φ, a term t admits at most one type. That is there is at most one type τ such that "$\Phi \vdash t : \tau$" is derivable. Moreover, in this case, the pair $(t : \tau)$ appears in the list Φ.

Proof. By induction on the derivation tree. □

Lemma 2.9 : Uniqueness of derivations

A given judgment admits at most one derivation tree.

Proof. At most one inference rule leads to each form of judgment. □

Lemma 2.10

The following rules are admissible.

$$\frac{\Psi \vdash \tau \text{ type} \qquad \Phi \vdash \alpha : \Psi}{\Phi \vdash \tau[\alpha] \text{ type}}\ \textsc{sub-typ} \qquad \frac{\Psi \vdash t : \tau \qquad \Phi \vdash \alpha : \Psi}{\Phi \vdash t[\alpha] : \tau[\alpha]}\ \textsc{sub-term}$$

$$\frac{\Phi \vdash \alpha : \Psi \qquad \Psi \vdash \beta : \Theta}{\Phi \vdash \beta \circ \alpha : \Theta}\ \textsc{sub-comp} \qquad \frac{\Phi \text{ ctx}}{\Phi \vdash \mathrm{id}_\Phi : \Phi}\ \textsc{sub-id}$$

Lemma 2.11

For any term t or type τ, when any of the following equation makes sense, it is satisfied.

$$t[\mathrm{id}_\Phi] = t \qquad t[\beta][\alpha] = t[\beta \circ \alpha]$$

$$\tau[\mathrm{id}_\Phi] = \tau \qquad \tau[\beta][\alpha] = \tau[\beta \circ \alpha]$$

The syntactic category

We will now define and characterise the syntactic category of $\textsc{PreTTy}_+$.

Definition 2.12 : Syntactic category

The syntactic category of the type theory $\textsc{PreTTy}_+$, denoted $\mathcal{S}_{\textsc{PreTTy}_+}$ is defined as follows.

- It has as objects the contexts Φ such that "Φ ctx" is derivable.
- It has as morphisms $\alpha : \Phi \to \Psi$ the substitutions α such that $\Phi \vdash \alpha : \Psi$ is derivable.

Remark 2.13
Note that it is a well-defined category thanks to Lemmas 2.10 and 2.11.

2.14 interpretation. We will now define an interpretation $[\![-]\!]$ of contexts, types and substitutions. This data will assemble as an equivalence of categories $[\![-]\!] : \mathcal{S}_{\textsc{PreTTy}_+} \to \hat{\mathcal{C}}_f$.

Definition 2.15 : Interpretation of contexts

Let Φ be a context, $[\![\Phi]\!] \in \hat{\mathcal{C}}_f$ is defined as follows.

$$\begin{aligned} [\![\Phi]\!]_c &= \{t \in \mathcal{A} \mid \Phi \vdash t : \tau \text{ holds for some } \tau \text{ with } \underline{\tau} = c\} \\ &= \{t \in \mathsf{FV}(\Phi) \mid (t, (c, -)) \in \Phi\} \end{aligned}$$

And, for any $t : \tau$ in $[\![\Phi]\!]_c$ and $u \in \mathcal{F}(c)$, $u^* t = \tau(u)$.

Definition 2.16 : Interpretation of types

Let τ be a type in some context Φ,

$$[\![\tau]\!]_\Phi : \partial\underline{\tau} \to [\![\Phi]\!]$$

is defined such that for any $(u : b \to \underline{\tau}) \in \mathcal{F}(\underline{\tau})$, $[\![\tau]\!]_{\Phi,b}(u) = \tau(u)$.

Remark 2.17
By factorizability of morphisms of $\mathcal{C}$ as maps in $\mathcal{F}$, the above determines completely the presheaf $[\![\Phi]\!]$ or the transformation $[\![\tau]\!]_\Phi$. However, we shall check that both are well-defined, this is done with the following lemma.

Lemma 2.18

For any context Φ (resp. type τ in a context Φ), its interpretation $[\![\Phi]\!]$ (resp. $[\![\tau]\!]_\Phi$) is well-defined. Moreover, the following holds:

(i) For a context Φ such that $\Phi \vdash x : \tau$ and $f : b \to \underline{\tau}$, $f^* x = [\![\tau]\!]_{\Phi,b}(f)$. That is, we have the following commutative diagram in $\hat{\mathcal{C}}_f$, where $c = \underline{\tau}$.

$$\begin{tikzcd} \partial c \arrow[r, hook] \arrow[dr, "{[\![\tau]\!]_\Phi}"'] & c \arrow[d, "t"] \\ & {[\![\Phi]\!]} \end{tikzcd}$$

(ii) For a type $\tau = (c, (t_u)_{u \in \mathcal{F}(c)})$ in a context Φ and $u : b_u \rightarrowtail c$, $[\![\tau]\!]_\Phi|_{b_u} = [\![\tau_u]\!]$ where $\Phi \vdash t_u : \tau_u$.

Definition 2.19 : Interpretation of substitutions

For any substitution $\alpha = \langle x_1 \mapsto t_1, \cdots, x_n \mapsto t_n\rangle$ such that $\Phi \vdash \alpha : \Psi$ is derivable, we let

$$[\![\alpha]\!] : [\![\Psi]\!] \to [\![\Phi]\!] \quad \text{be defined by} \quad [\![\alpha]\!](x_i) = t_i$$

That is, we have the equations $[\![\alpha]\!](x) = x[\alpha]$.

Lemma 2.20

Definition 2.19 yields a natural transformation, preserves identities and reverse compositions.

Proof. Every morphism $f : z \to c$ in $\mathcal{C}$ factors as a finite composition of maps u_i of $\mathcal{F}$:

$$f = u_n \circ \cdots \circ u_1$$

for some $n \in \mathbb{N}$. So it suffices to show that $[\![\alpha]\!]$ commutes with the u_i's.

Let $x \in [\![\Psi]\!]_c$ for some $c \in \mathcal{O}b(\mathcal{C})$. Then $\Psi \vdash x : \tau$ is derivable for some τ with $\underline{\tau} = c$. Hence, using Lemma 2.10, we have $\Phi \vdash t[\alpha] : \tau[\alpha]$ derivable, so $t[\alpha] = [\![\alpha]\!](t) \in [\![\Psi]\!]_c$. Moreover, for any $u \in \mathcal{F}(c)$, $u^*(t[\alpha]) = \tau[\alpha](u) = \tau(u)[\alpha] = (u^* t)[\alpha]$, whence $[\![\alpha]\!](u^* t) = u^*([\![\alpha]\!](t))$. Which yields the naturality.

We see by definition that identity substitution is sent to identity transformation and that composites of substitutions are sent to the reverse composites of transformations. $\square$

Definition 2.21 : $[\![-]\!] : \mathcal{S}_{\text{PRETTY}} \to \hat{\mathcal{C}}_f^{\text{op}}$

The interpretation of contexts and substitutions as given by Definitions 2.15 and 2.19 yields a contravariant functor from the syntactic category to the category of finite presheaves on $\mathcal{C}$, which we denote $[\![-]\!]$ and call the *interpretation* or *semantic* functor.

Lemma 2.22

Let $\alpha : \partial c \to [\![\Phi]\!]$ for some context Φ, then $\alpha = [\![\tau_\alpha]\!]_\Phi$ for some unique well-formed type τ_α in the context Φ.

Lemma 2.23

Let Φ be a context and $\Psi = (\Phi, x : \tau)$ obtained by ctx-EXT. Then $[\![\Psi]\!]$ is the following pushout:

$$\begin{tikzcd}
\partial c \arrow[r, "{[\![\tau]\!]_\Phi}"] \arrow[d, hook] & {[\![\Phi]\!]} \arrow[d, hook] \\
c \arrow[r, "x"'] & {[\![\Psi]\!]} \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

Proof. The commutatitvity of the square above is by definition of $[\![\Psi]\!]$. Let Z be a presheaf over $\mathcal{C}$ and $\alpha : [\![\Phi]\!] \to Z$, $z \in Z_c$ such that the following square commutes:

$$\begin{tikzcd}
\partial c \arrow[r, "{[\![\tau]\!]_\Phi}"] \arrow[d, hook] & {[\![\Phi]\!]} \arrow[d, "\alpha"] \\
c \arrow[r, "z"'] & Z
\end{tikzcd}$$

If α and z factor through some $\beta : [\![\Psi]\!] \to Z$, then β is completely determined as a function by $\beta|_{[\![\Phi]\!]} = \alpha$ and $\beta(x) = z$. In order to see that β defined as such is natural, we need to check that for any morphism $f : a \to c$, $\alpha(f^*x) = f^*z$. Assuming $f \neq \mathsf{id}$, this is given by the assumption $\alpha \circ [\![\tau]\!]_\Phi = z$ together with the point (i) of Lemma 2.18. $\square$

Theorem 2.24

$[\![-]\!] : \mathcal{S}_{\text{PRETTY}_+} \longrightarrow \hat{\mathcal{C}}_f^{\text{op}}$ is an equivalence of categories.

Proof. We need to check that $[\![-]\!]$ is fully faithfull and essentially surjective. We fix two contexts Φ and Ψ.

- *faithfullness.* Let $\varphi : [\![\Psi]\!] \to [\![\Phi]\!]$ be a natural transformation. Suppose $\varphi = [\![\alpha]\!]$ for some substitution α. Then α must be of the form $\langle x \mapsto \varphi(x)\rangle_{x \in \mathsf{FV}(\Psi)}$, whence the faithfullness.
- *fullness.* Let $\varphi : [\![\Psi]\!] \to [\![\Phi]\!]$ be a natural transformation and let $\alpha = \langle x \mapsto \varphi(x)\rangle_{x \in \mathsf{FV}(\Psi)}$. For any $t \in [\![\Psi]\!]_c$, t is a variable in $\mathsf{FV}(\Psi)$ according to Lemma 2.8. Hence $[\![\alpha]\!](t) = \varphi(t)$, whence $\varphi = [\![\alpha]\!]$.
- *essential surjectivity.* Let $X \in \hat{\mathcal{C}}_f$. According to Remark 1.14, X admits a finite number of elements x_i $(1 \leq i \leq n)$. We proceed by induction on n.
  - Suppose $n = 0$, then X is the empty presheaf, and is the image of the empty context.

– Suppose $n > 0$. Suppose $x = x_n \in X_c$ is maximal in the sense that it may not be written as $f^* y$ for some other cell y of X. Let $Y := X \setminus \{x\}$. Since x is maximal, we may check that Y is again a (finite) presheaf. There is an inclusion $Y \hookrightarrow X$ and a map $c \xrightarrow{x} X$ given by the YONEDA embedding. We see that those two maps exhibit X as the following pushout:

$$\begin{tikzcd} \partial c \arrow[r, "x|_{\partial c}"'] \arrow[d, hook] & Y \arrow[d, hook] \\ c \arrow[r, "x"] & X \arrow[ul, phantom, "\ulcorner" very near start] \end{tikzcd}$$

Let $z : c \to Z$ (seen as $z \in Z_c$) and $\varphi : Y \to Z$ such that $\varphi \circ x|_{\partial c} = z|_{\partial c}$. If, φ and z factor through $\psi : X \to Z$, then $\psi(x) = z$ and $\psi|_Y = \varphi : Y \to Z$, so ψ is completely defined as a natural transformation $X \to Z$.
In order to see that ψ defined as such is a well-defined natural transformation, we need to see that for any $f : z \to c$, $\psi(f^* x) = f^* \psi(x)$. Assuming $f \neq \mathsf{id}$, this equation is precisely given by the assumption $\varphi \circ x|_{\partial c} = z|_{\partial c}$.
Now, using the induction hypothesis with Y yields a context Φ such that $[\![\Phi]\!] \simeq Y$, and Lemma 2.22 gives a type τ in Φ such that $[\![\tau]\!] = x|_{\partial c}$. Then Lemma 2.23 proves that $X \simeq [\![\Phi, x : \tau]\!]$, where $\Phi, x : \tau$ is obtained by CTX-EXT from Φ. □

3 Presheaves over REEDY categories

3.1 Assumptions. In this section we fix $\mathcal{C}$ an elegant REEDY category with two-layered boundaries, indexed over the ordinal ω, with two wide subcategories $\mathcal{C}_+$ and $\mathcal{C}_-$ as classes of upward and downard maps. We moreover assume that :

- The category $\mathcal{C}_+$ is locally finite and monic.
- We are given a finite presentation P of $\mathcal{C}_-$ which does not contains identities.

Our aim is to define an extension of PRETTY$_+$, which we call PRETTY (or PRETTY$_\mathcal{C}$), whose syntactic categories consists of finitety generated presheaves over $\mathcal{C}$, and whose models are the presheaves over $\mathcal{C}$.
We stick to the notations $\mathcal{F}$, $\mathcal{J}_{u,v}$, $p_{i,j}(u, v)$ and $a_j(u, v)$ introduced in Section 2. For any composable pair

$$c' \xrightarrow{v \in \mathcal{F}(d)} d \xrightarrow{p \in P} c$$

such that $p \circ v \neq \mathsf{id}$, we will assume the unique factorisation of $p \circ v$ in $(\mathcal{C}_-, \mathcal{C}_+)$ to be given by

$$c' \xrightarrow{q(p,v) \in P} b \xrightarrow{w(p,v) \in \mathcal{F}(c)} c \qquad .$$

The type theory

We first define the theory PRETTY, an extension of the theory PRETTY$_+$. If we need to precise the category $\mathcal{C}$ we are working with, we will write PRETTY$_\mathcal{C}$.

3.2 Terms. We consider the type theory for presheaves over $\mathcal{C}_+$, which we denote PRETTY$_+$ and extend it to the type theory PRETTY by adding formal degenerescences to variables. The terms of PRETTY are given by the following grammar:

$$\begin{array}{rcll} \mathsf{tm} & ::= & x & (x \in \mathcal{A}) \\ & | & p^*\mathsf{tm} & (p \in P) \end{array}$$

where $\mathcal{A}$ still denotes a denumerable set of variables as assumed in Subsection 2. We denote their set by tm.

3.3 Syntax. As the theory PRETTY$_+$, PRETTY has *terms*, *types*, *contexts* and *substitutions*. The terms are given by 3.3. The syntax of types, contexts, and substitutions remains unchanged from PRETTY$_+$.

3.4 Judgments. In addition to the four kind of judgment introduced for PRETTY$_+$, we add the following kinds.

- The judgment "$\Phi \vdash t \equiv u$" expresses that in the context Φ, the terms t and u are semantically equals.
- The judgment "$\Phi \vdash \tau \equiv \sigma$ type" expresses that in the context Φ, the types τ and σ are semantically equals.
- The judgment "$\vdash \Phi \equiv \Psi$ ctx" expresses that the contexts Φ and Ψ are semantically equals.

3.5 Free variables. The free variables of a term are now given by:

- $\mathsf{FV}(x) = \{x\}$ for $x \in \mathcal{A}$.
- $\mathsf{FV}(p^*t) = \mathsf{FV}\{t\}$ for t a term and $p \in P$.

The definition of free variables for types, contexts, and substitution remains unchanged from PRETTY$_+$.

3.6 Substitutions in terms and types. We define the action of a substitution $\alpha = \langle x_i \mapsto t_i \rangle_i$ in a term as follows.

- For x a variable:
$$x[\alpha] = \begin{cases} t_i & \text{if } x = x_i \text{ for some } i. \\ x & \text{in the other cases.} \end{cases}$$
- For t a term and $p \in P$:
$$(p^*t)[\alpha] = p^*(t[\alpha])$$

Substitutions act on types exactly as they do in PRETTY$_+$ (*c.f.* 2.4).

3.7 Action of degeneracies on types. Let $(p : d \to c) \in P$, t a term, and $\tau = (c, (t_u)_{u \in \mathcal{F}(c)})$ a type. Then we define a type $(p, t)^*\tau = (d, (s_v)_{v \in \mathcal{F}(d)})$, where

$$s_v = \begin{cases} t & \text{if } p \circ v = \mathrm{id}_c. \\ q(p, v)^* t_{w(p,v)} & \text{in the other cases.} \end{cases}$$

Notice that when it makes sense, the following equation holds:

$$((p, t)^*\tau)[\alpha] = (p, t)^*(\tau[\alpha])$$

3.8 Inference rules. In addition to the rules already introduced for PRETTY$_+$, we add the following ones:

$$\frac{\Phi \vdash t : \tau \qquad (p : - \to \underline{\tau}) \in P}{\Phi \vdash p^*t : (p, t)^*\tau}\ \text{DEGE} \qquad\qquad \frac{\Phi \vdash t : \tau \qquad (p_1 \circ \cdots \circ p_n) P (q_1 \circ \cdots \circ q_m)}{\Phi \vdash p_n^* \cdots p_1^* t \equiv q_m^* \cdots q_1^* t}\ \text{REL}$$

degeneracy rules

$$\frac{\Phi \vdash t : \tau}{\Phi \vdash t \equiv t}\ \text{REFL} \qquad \frac{\Phi \vdash t \equiv u}{\Phi \vdash u \equiv t}\ \text{SYM} \qquad \frac{\Phi \vdash t \equiv u \qquad \Phi \vdash u \equiv v}{\Phi \vdash t \equiv v}\ \text{TRANS}$$

$$\frac{\Phi \vdash t : \tau \qquad \Phi \vdash u : \tau \qquad \Phi \vdash t \equiv u \qquad (p : - \to \underline{\tau}) \in P}{\Phi \vdash p^*t \equiv p^*u}\ \text{EQUIV}$$

$$\frac{\Phi \vdash \tau \text{ type} \qquad \Phi \vdash \sigma \text{ type} \qquad \bigwedge_{u \in \mathcal{F}(c)} \Phi \vdash \tau(u) \equiv \sigma(u)}{\Phi \vdash \tau \equiv \sigma \text{ type}}\ \text{TYPE-EQ} \quad \underline{\tau} = \underline{\sigma} = c$$

$$\frac{\vdash \Phi \equiv \Psi \text{ ctx} \qquad \Phi \vdash \tau \equiv \sigma \text{ type}}{\vdash (\Phi, x : \tau) \equiv (\Psi, x : \sigma) \text{ ctx}}\ \text{CTX-EQ} \quad x \notin \mathsf{FV}(\Phi) \cap \mathsf{FV}(\Psi)$$

equality rules

Some properties of PRETTY

The following lemmas may be proven by induction on the derivation trees.

Lemma 3.9

The following properties may be shown by induction on the derivation trees.

- If "$\Phi \vdash \tau$ type" is derivable, so is "Φ ctx" and $\mathsf{FV}(\tau) \subseteq \mathsf{FV}(\Phi)$.
- If "$\Phi \vdash t : \tau$" is derivable, so is "$\Phi \vdash \tau$ type" and $\mathsf{FV}(t) \subseteq \mathsf{FV}(\Phi)$.
- If "$\Phi \vdash \alpha : \Psi$" is derivable, so are "$\Phi$ ctx" and "Ψ ctx".
- If "$\Phi \vdash (c, (t_u)_{u \in \mathcal{F}(c)})$ type" is derivables, all terms t_u are typeable is the context Φ.
- If "$\Phi \vdash t \equiv s$" is derivable, then there are types τ, σ such that $\Phi \vdash t : \tau$, $\Phi \vdash s : \tau$ and $\Phi \vdash \tau \equiv \sigma$ holds.
- If "$\vdash \Phi \equiv \Psi$ ctx" is derivable, then a judgment "$\Phi \vdash \tau$ type" holds iff "$\Psi \vdash \tau$ type" holds. And similarly, "$\Phi \vdash t : \tau$" holds iff "$\Psi \vdash t : \tau$" holds.

Lemma 3.10 : Uniqueness of type

In a given context Φ, a term t admits at most one type up to $\equiv$.

Lemma 3.11

A judgment $\Phi \vdash t : \tau$ holds iff there is a sequence of composable elements $p_1, \cdots, p_n \in P$, a variable $x \in \mathsf{FV}(\Phi)$ and a type σ such that

$$t = p_n^* \cdots p_1^* x \qquad \tau = (p_n, p_{n-1}^* \cdots p_1^* x)^* \cdots (p_1, x)^* \sigma \qquad \text{with } (x : \sigma) \in \Phi.$$

Lemma 3.12

$\Phi \vdash t \equiv s$ holds iff there are sequences $p_1, \cdots p_n \in P$ and $q_1, \cdots q_m \in P$ and a variable $x \in \mathsf{FV}(\Phi)$ such that

$$t = p_n^* \cdots p_1^* x \qquad s = q_m^* \cdots q_1^* x \qquad \text{with } (p_1 \circ \cdot \circ p_n) P (q_1 \circ \cdots q_m)$$

Hence we denote unambiguously $f^* x$ for the class of $p_n^* \cdots p_1^* x$ in $\mathcal{T}m/_{\equiv}$ given by any choice of decomposition $f = p_1 \circ \cdots \circ p_n$ with $p_i \in P$ for all i's.

The syntactic category

3.13 As for the direct case, the rules exposed in Lemma 2.10 are still admissible. Moreover, notice that every valid derivation tree in PRETTY$_+$ remains valid in PRETTY, hence every context (resp. substitution) in PRETTY$_+$ is a context (resp. substitution) in PRETTY.

Definition 3.14 : $\mathcal{S}_{\text{PRETTY}}$

We let $\mathcal{S}_{\text{PRETTY}}$ denotes the category whose objects are the well-formed contexts of PRETTY, and morphisms are the well-defined substitutions up to $\equiv$. According to the previous assertion, there is a functor $\mathcal{S}_{\text{PRETTY}_+} \to \mathcal{S}_{\text{PRETTY}}$ which is the identity on objects and morphisms.

Definition 3.15 : Interpretation of contexts

For Φ ctx a well-formed context of PRETTY, its *interpretation* is the finitely generated presheaf $[\![\Phi]\!] \in \hat{\mathcal{C}}_f$ defined by

$$[\![\Phi]\!]_c = \{t : \tau \mid \Phi \vdash t : \tau \text{ is provable and } \underline{\tau} = c\} / \equiv$$

And, for any $t : \tau$ in $[\![\Phi]\!]_c$ and $u \in \mathcal{F}(c)$, $[\![\Phi]\!](u)(t) = \tau(u)$, for any $p \in P$, $[\![\Phi]\!](p)(x) = p^*x$.

Definition 3.16 : Interpretation of types

For $\Phi \vdash \tau$ type a derivable judgment, the *interpretation* of τ is the natural transformation

$$[\![\tau]\!]_\Phi : \partial\underline{\tau} \to [\![\Phi]\!]$$

such that for $(u : b \to \underline{\tau}) \in \mathcal{F}(\underline{\tau})$, $[\![\tau]\!]_{\Phi,b}(u) = \tau(u)$.

Remark 3.17
By factorizability of morphisms of $\mathcal{C}$ as maps in $\mathcal{F}$ and P, Definition 3.15 above determines completely the presheaf $[\![\Phi]\!]$. On the other hand, since any element of $\partial\underline{\tau}$ factorizes by some element $u \in \mathcal{F}(\underline{\tau})$, the transformation $[\![\tau]\!]_\Phi$ is also fully determined by the specification of Definition 2.16. However, we shall check that both are well-defined, this is done with the following lemma.

Lemma 3.18

For any context Φ (resp. type τ in a context Φ), its interpretation $[\![\Phi]\!]$ (resp. $[\![\tau]\!]_\Phi$) is well-defined. Moreover, the following holds:

(i) For a context Φ such that $\Phi \vdash x : \tau$ and $f : b \to \underline{\tau}$, $f^*x = [\![\tau]\!]_{\Phi,b}(f)$. That is, we have the following commutative diagram in $\hat{\mathcal{C}}_f$, where $c = \underline{\tau}$.

$$\begin{tikzcd} \partial c \arrow[r, hook] \arrow[dr, "{[\![\tau]\!]_\Phi}"'] & c \arrow[d, "t"] \\ & {[\![\Phi]\!]} \end{tikzcd}$$

(ii) For a type $\tau = (c, (t_u)_{u \in \mathcal{F}(c)})$ in a context Φ and $u : b_u \rightarrowtail c$, $[\![\tau]\!]_\Phi|_{b_u} = [\![\tau_u]\!]$ where $\Phi \vdash t_u : \tau_u$.

Proof. We proceed by induction on derivation trees to prove those properties.

- (*well-definedness of* $[\![\Phi]\!]$) Suppose that "Φ ctx" is derivable.
  - Either $\Phi = \varnothing$ is the empty context and $[\![\Phi]\!] = \varnothing$ is well-defined as the empty presheaf.
  - Or $\Phi = (\Psi, x : \tau)$ is obtained by context extension (using CTX-EXT) of Ψ. Let $c = \underline{\tau}$. In this case, $[\![\Psi]\!]$ and $[\![\tau]\!]_\Psi : \partial c \to [\![\Psi]\!]$ are already defined. Hence $[\![\Phi]\!]$ may be described as the following pushout.

$$\begin{tikzcd} \partial c \arrow[r, "{[\![\tau]\!]_\Psi}"] \arrow[d, hook] & {[\![\Psi]\!]} \arrow[d, hook] \\ c \arrow[r, "x"'] & {[\![\Phi]\!]} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

- (*well-definedness of* $[\![\tau]\!]_\Phi$) Suppose that "$\Phi \vdash \tau$ type" is derivable. Then the following premises of rule TYPE holds.

$$\Phi \text{ ctx} \qquad \bigwedge_{u \in \mathcal{F}(c)} \Phi \vdash t_u : \tau_u \qquad \bigwedge_{\substack{u_1 \neq u_2 \in \mathcal{F}(c) \\ j \in \mathcal{J}_{u_1,u_2}}} \tau_{u_1}(p_{1,j}) = \tau_{u_2}(p_{2,j})$$

Suppose given two distinct factorizations $f = u \circ g = v \circ h$ of some element $f \in (\partial c)_z$ where $\underline{\tau} = c$ and $u : b_u \rightarrowtail c$, $v : b_v \rightarrowtail c$. By monicness of u, v it must be the case that $u \neq v$. Then the universal property of the pullback $a(u, v)$ yield a commutative diagram in $\hat{\mathcal{C}}$.

$$
\begin{tikzcd}
 & b_u \arrow[r, tail, "u"] & c \\
 & a \arrow[u, "p_1"] \arrow[r, "p_2"'] \arrow[ur, phantom, "\urcorner"] & b_v \arrow[u, tail, "v"'] \\
z \arrow[uur, bend left, "g"] \arrow[ur, dashed, "k"] \arrow[urr, bend right, "h"'] & &
\end{tikzcd}
$$

Since $z \in \mathcal{C}$, $k : z \to a$ must factor as $k_j : z \to a_j$ through some $a_j \hookrightarrow a$, for some $j \in \mathcal{J}_{u,v}$. Hence, the assumption $\tau_u(p_{1,j}) = \tau_v(p_{2,j})$ allows us to write

$$g^* t_u = k_j^* p_{1,j}^* \, t_u = k_j^* \tau_u(p_{1,j}) = k^* \tau_v(p_{2,j}) = k_j^* p_{2,j}^* \, t_v = g^* t_v \quad .$$

We then see that $[\![\tau]\!]_\Phi$ is natural: if $f_1 : a_1 \to c$ equals $f_2 \circ g$ for f_1 and $f_2 : a_2 \to c$ two elements of ∂c, then picking a factorization $f_2 = u \circ h$ for some $u \in \mathcal{F}(c)$ yields

$$[\![\tau]\!]_\Phi(f_1) = [\![\tau_u]\!]_\Phi(g \circ h) = g^*[\![\tau_u]\!]_\Phi(h) = g^*[\![\tau]\!]_\Phi(f_2) \quad .$$

(i) Suppose that $\Phi \vdash x : \tau$ is derivable and let $f : b \to c$ where $c = \underline{\tau}$. Let $f = u \circ g$ be a factorization where $u \in \mathcal{F}$. Then

$$f^* x = g^* u^* x = g^* \tau(u) = [\![\tau]\!]_\Phi(u \circ g) = [\![\tau]\!]_{\Phi, b}(f) \quad .$$

Where we have used the definition of $[\![\Phi]\!]$ and the naturality of $[\![\tau]\!]$.

(ii) Suppose $\tau = (c, (t_u)_{u \in \mathcal{F}(c)})$ is such that "$\Phi \vdash \tau$ type" is derivable. Let $u : b_u \rightarrowtail c$ and suppose $t_u : \tau_u$. Then for any $f : z \to b_u$, we have

$$[\![\tau]\!]_z(u \circ f) = f^*[\![\tau]\!]_{b_u}(u) = f^* \tau(u) = [\![\tau_u]\!]_\Phi(f) \quad .$$

Where we have used the naturality of $[\![\tau]\!]_\Phi$ and the point (i). ☐

Definition 3.19

For any substitution $\alpha = \langle x_1 \mapsto t_1, \cdots, x_n \mapsto t_n \rangle$ such that $\Phi \vdash \alpha : \Psi$ is derivable, we let

$$[\![\alpha]\!] : [\![\Psi]\!] \to [\![\Phi]\!] \quad \text{be defined by} \quad [\![\alpha]\!](f^* x_i) = f^* t_i$$

That is, we have the equations $[\![\alpha]\!](t) = t[\alpha]$.

Lemma 3.20

Definition 3.19 yields a natural transformation, preserves identities and reverse compositions.

Proof. Let α be a substitution from Φ to Ψ. Let $t \in [\![\Psi]\!]_c$ for some $c \in \mathcal{O}b(\mathcal{C})$. Then $\Psi \vdash t : \tau$ is derivable for some τ with $\underline{\tau} = c$. Hence as mentionned in 3.13, we have $\Phi \vdash t[\alpha] : \tau[\alpha]$ derivable, whence $t[\alpha] = [\![\alpha]\!](t) \in [\![\Psi]\!]_c$. Hence $[\![\alpha]\!]$ is well-defined.

Now we check the naturality. By assumptions on $\mathcal{C}$, any morphism is a composition of maps $u \in \mathcal{F}(c)$ for some c and $p \in P$. So it suffices to show that $[\![\alpha]\!]$ commutes with the u's and the p's.

- for any $u \in \mathcal{F}(c)$, $[\![\Phi]\!](u)(t[\alpha]) = \tau[\alpha](u) = \tau(u)[\alpha] = ([\![\Phi]\!](u)(t))[\alpha]$.
- for any $p \in P$, $[\![\Phi]\!](p)(t[\alpha]) = p^*(t[\alpha]) = (p^* t)[\alpha] = ([\![\Phi]\!](p)(t))[\alpha]$.

Whence the naturality.

We see by definition that identity substitution is sent to identity transformation and that composites of substitutions are sent to the reverse composites of transformations. ☐

Definition 3.21 : $[\![-]\!] : \mathcal{S}_{\text{PRETTY}} \to \hat{\mathcal{C}}_f^{\text{op}}$

The interpretation of contexts and substitutions as given by Definitions 3.15 and 3.19 gives a contravariant functor from the syntactic category to the category of finite presheaves on $\mathcal{C}$, which we denote $[\![-]\!]$ and call the *interpretation* or *semantic* functor.

Lemma 3.22

Let $\alpha : \partial c \to [\![\Phi]\!]$ for some context Φ, then $\alpha = [\![\tau_\alpha]\!]$ for some unique well-formed type τ_α up to $\equiv$ in the context Φ.

Proof. We proceed by induction on the object $c \in \mathcal{O}b(\mathcal{C})$, using the well-foundedness of $\lhd$. Using the universal property of ∂c given by Proposition 1.18, α restricts to maps $\alpha_u : b_u \to [\![\Phi]\!]$ for all $(u : b_u \to c) \in \mathcal{F}(c)$, which corresponds to terms $(\alpha(u) = t_u : \tau_u) \in [\![\Phi]\!]_{b_u}$ according to Definition 3.15, for t_u, τ_u defined up to $\equiv$. By Lemma 3.18, $[\![\tau_u]\!]_\Phi$ is the restriction of t_u along its boundary $\partial b_u \to [\![\Phi]\!]$ for each u. Moreover, still by definition of ∂c, those maps α_{u_1}, α_{u_2} (with $u_1 \neq u_2$) must coincide when restricted along the legs p_1, p_2 of the associated pullback. According to Definition 3.15, it gives the equations $\tau_{u_1}(p_{1,j}) = \tau_{u_2}(p_{2,j})$ for $j \in \mathcal{J}_{u_1,u_2}$. Hence the type $\tau = (c, (t_u)_{u \in \mathcal{F}(c)})$ is well formed in the context Φ. Now, by Definition 3.16 and Lemma 3.18, $[\![\tau]\!]_\Phi = \alpha$ and τ as described above was the only possible type up to $\equiv$. $\square$

Lemma 3.23

Let Φ be a context and $\Psi = (\Phi, x : \tau)$ obtained by CTX-EXT. Then $[\![\Psi]\!]$ is the following pushout, where $\pi : \Psi \to \Phi$ is the canonical substitution:

$$\begin{tikzcd}
\partial c \arrow[r, "{[\![\tau]\!]_\Phi}"] \arrow[d] & {[\![\Phi]\!]} \arrow[d, "{[\![\pi]\!]}"] \\
c \arrow[r, "x"'] \arrow[ur, phantom, "\ulcorner"] & {[\![\Psi]\!]}
\end{tikzcd}$$

Proof. The commutatitvity of the square above is by definition of $[\![\Psi]\!]$. Let Z be a presheaf over $\mathcal{C}$ and $\alpha : [\![\Phi]\!] \to Z$, $z \in Z_c$ such that the following square commutes:

$$\begin{tikzcd}
\partial c \arrow[r, "\tau"] \arrow[d] & {[\![\Phi]\!]} \arrow[d, "\alpha"] \\
c \arrow[r, "z"'] & Z
\end{tikzcd}$$

If α and z factor through some $\beta : [\![\Psi]\!] \to Z$, then β is completely defined as a function by

- $\beta(t) = \alpha(t)$ if $t = p^* y$ for some $p \in \mathcal{C}_-$ and $y \in \mathsf{FV}(\Phi)$.
- $\beta(p^* x) = p^* z$ for $(p : - \to c) \in \mathcal{C}_-$.

Now, we check that β is well-defined as a natural transformation.

- If $t \in [\![\Phi]\!]_b$ and $(f : - \to b) \in \mathcal{C}$, $\beta(f^* t) = \alpha(f^* t) = f^* \alpha(t) = f^* \beta(t)$.
- If $t = x$ and $(f : - \to c) \in \mathcal{C}$, such that $f = f_+ \circ f_-$ with $f_+ \in \mathcal{C}_+$ and $f_- \in \mathcal{C}_-$, then $\beta(f^* x) = \beta(f_-^* f_+^* x)$
  - Either $f_+ = \mathsf{id}$, hence $f_- = f$ and $\beta(f^* x) = f^* \beta(x)$ by definition
  - Or $f_+ \neq \mathsf{id}$, hence $f_-^* f_+^* x \in [\![\Phi]\!]$ and $\beta(f_-^* f_+^* x) = \alpha(f_-^* f_+^* x) = f_-^* \alpha(f_+^* x)$. Then $\alpha(f_+^* x) = f_+^* z = f_+^* \beta(x)$ because the above square is commutative, whence $\beta(f^* x) = f^* \beta(x)$.

- If $t = p^*x$ for some $p \in \mathcal{C}_-$, then the previous point shows that for any f, $\beta(f^*p^*x) = f^*p^*\beta(x) = f^*\beta(p^*x)$, whence $\beta(f^*t) = f^*\beta(t)$.

This proves the naturality condition, hence it show that $[\![\Psi]\!]$ satisfies the universal property of the pushout. □

Theorem 3.24

$[\![-]\!]$ is an equivalence of categories.

Proof. We need to check that $[\![-]\!]$ is fully faithfull and essentially surjective. We fix two contexts Φ and Ψ of PreTTy.

- *faithfullness.* Let $\varphi : [\![\Psi]\!] \to [\![\Phi]\!]$ be a natural transformation. Suppose $\varphi = [\![\alpha]\!]$ for some substitution α. Then α must be of the form $\langle x \mapsto \varphi(x)\rangle_{x \in \mathsf{FV}(\Psi)}$ (which is well-defined, up to $\equiv$), whence the faithfullness.
- *fullness.* Let $\varphi : [\![\Psi]\!] \to [\![\Phi]\!]$ be a natural transformation and let $\alpha = \langle x \mapsto \varphi(x)\rangle_{x \in \mathsf{FV}(\Psi)}$. For any $t \in [\![\Psi]\!]_c$, t is of the form f^*x for some $f \in \mathcal{C}_-$. By naturality, $\varphi(t) = \varphi([\![\Phi]\!](f)(x)) = [\![\Phi]\!](f)(\varphi(x)) = f^*\varphi(x)$. Hence $\varphi = [\![\alpha]\!]$.
- *essential surjectivity.* Let $X \in \hat{\mathcal{C}}_f$. According to Lemma 1.28, X admits a finite number of generators x_i $(1 \leq i \leq n)$. We proceed by induction on n.
  - Suppose $n = 0$, then X is the empty presheaf, and is the image of the empty context.
  - Suppose $n > 0$. Suppose $x = x_n \in X_c$ is of maximal dimension. Let $Y := X \setminus \{f^*x\}_{f \in \mathcal{C}_-}$. That is, Y is X without the degeneracies of x. Since $\mathcal{C}$ is eleguant, we may check that Y is again a presheaf. Moreover, the non-degenerated cells of Y are exactly the $\{x_i\}_{1 \leq i < n}$, hence Theorem 1.29 shows that $Y \in \hat{\mathcal{C}}_f$. There is an inclusion $Y \hookrightarrow X$ and a map $c \xrightarrow{x} X$ given by the Yoneda embedding. We see that those two maps make X the following pushout:

$$\begin{tikzcd} \partial c \arrow[r, "x|_{\partial c}"'] \arrow[d] & Y \arrow[d, hook] \\ c \arrow[r, "x"] & X \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

    Let $z : c \to Z$ (seen as $z \in Z_c$) and $\varphi : Y \to Z$ such that $\varphi \circ x|_{\partial c} = z|_{\partial c}$. If, φ and z factor through $\psi : X \to Z$, then $\psi(x) = z$ and $\psi|_Y = \varphi : Y \to Z$, so ψ is completely defined as a natural transformation $X \to Z$ by $\psi(f^*x) = f^*z$ for $f \in \mathcal{C}_-$.
    We then check that ψ is natural. Let $x' \in X$ and g such that g^*x' makes sense. Because φ is natural, if $x' \in Y$, we already have $\psi(g^*x') = g^*\psi(x')$. For $x' = x$, write $g = g_+ \circ g_-$ for some $g_+ \in \mathcal{C}_+$ and $g_- \in \mathcal{C}_-$. If $g = g_-$, $\psi(g^*x) = g^*\psi(x)$ is by definition. Assume $g_+ \neq \mathrm{id}$.

$$\begin{array}{rcll} \psi(g^*x') & = & \psi(g_-^*g_+^*x) & \\ & = & g_-^*\psi(g_+^*x) & \text{because } g_+^*x \in Y \\ & = & g_-^*g_+^*\psi(x) & \text{because } \varphi \circ x|_{\partial c} = z|_{\partial c} \\ & = & g^*\psi(x) & \end{array}$$

    Then, if $x' = f^*x$ for some $f \in \mathcal{C}_-$ and $g \in \mathcal{C}$, the above property yield $\psi(g^*x') = g^*f^*\psi(x') = g^*\psi(f^*x')$ whence the result.
    Now, using the induction hypothesis with Y yields a context Φ such that $[\![\Phi]\!] \simeq Y$, and Lemma 3.22 gives a type τ in Φ such that $[\![\tau]\!] = x|_{\partial c}$. Then Lemma 3.23 proves that $X \simeq [\![\Phi, x : \tau]\!]$, where $\Phi, x : \tau$ is obtained by ctx-ext from Φ. □

4 Set-valued models of PreTTy

In this section, our aim is to characterize the (set-valued) models of PreTTy$_{\mathcal{C}}$. As expected, we will see that they corresponds precisely to presheaves over $\mathcal{C}$. We start by noticing that the syntactic category of PreTTy admits a structure of *category with families* (CwF). We refer the reader to [4] for a gentle introduction to this notion.

Definition 4.1

From now on, $\mathcal{S}_{\textsc{PreTTy}}$ will be seen as the following CwF.

- $\mathcal{S}_{\textsc{PreTTy}}$ is the underlying category.
- For a context Φ, $\mathcal{T}y^{\Phi} = \{\tau \mid \Phi \vdash \tau \text{ type is derivable}\}$.
- For a context Φ and $\tau \in \mathcal{T}y^{\Phi}$, $\mathcal{T}m^{\Phi}_{\tau} = \{t \mid \Phi \vdash t : \tau \text{ is derivable}\}$.
- The values of $(\mathcal{T}y, \mathcal{T}m)$ on substitutions is given by the action of substitutions on types and terms.
- The specified terminal object of $\mathcal{S}_{\textsc{PreTTy}}$ is the empty context $\emptyset$.
- For a context Φ and $\tau \in \mathcal{T}y^{\Phi}$, the context comprehension operation is given by:
  - The object $(\Phi, a : \tau)$ obtained by the rule ctx-EXT.
  - The substitution $\pi = \langle x \mapsto x\rangle_{x \in \mathsf{FV}(\Phi)} : (\Phi, a : \tau) \to \Phi$. where a is minimal in $\mathcal{A} \setminus \mathsf{FV}(\Phi)$.
  - The term $a \in \mathcal{T}m^{(\Phi, a:\tau)}_{\tau}$ found as said above.

4.2 For any object c of $\mathcal{C}$, the presheaf $\mathsf{hom}(-, c)$ and representing c its boundary ∂c are finitely generated (*c.f.* Remark 1.14). Hence, using the equivalence $\mathcal{S}_{\textsc{PreTTy}} \simeq \hat{\mathcal{C}}_f$ of Theorem 3.24, we let Φ_c (resp. $\Phi_{\partial c}$) denotes a context such that $[\![\Phi_c]\!] \simeq \mathsf{hom}(-, c)$ (resp. $[\![\Phi_{\partial c}]\!] \simeq \partial c$).

By definition of $[\![]\!]$, for any object c and any $(u : b_u \to c) \in \mathcal{F}(c)$, the judgment

$$\Phi_{\partial c} \vdash e(u) : (b_u, (e(v \circ u))_{v \in \mathcal{F}(b_u)})$$

is derivable, where $e : \mathsf{hom}_{\mathcal{C}_-}(-, c) \to \mathcal{A}$ is an encoding of every $f \in \mathcal{C}_-(-, c)$ as a variable $e(f) \in \mathcal{A}$. Then, using the rule TYPE, the judgment

$$\Phi_{\partial c} \vdash \sigma_c \text{ type}$$

holds, where $\sigma_c = (c, e(u)_{u \in \mathcal{F}(c)})$. So Φ_c is obtained by a context extension (using CTX-EXT) from $\Phi_{\partial c}$. In particular, there is an isomorphism α_c and a display map π_c as follows:

$$\begin{tikzcd}
\Phi_c \arrow[r, "\sim", "\alpha_c"'] \arrow[dr, dashed, "\xi_c"'] & (\Phi_{\partial c}, a : \sigma_c) \arrow[d, "\pi_c"] \\
& \Phi_{\partial c}
\end{tikzcd}$$

such that α_c, π_c and $\xi_c = \alpha_c \circ \pi_c$ are natural in c.

Lemma 4.3 : Representability of types and terms

For any object $c \in \mathcal{C}$ and context Ψ of $\textsc{PreTTy}$, the map

$$\begin{array}{rcl} \mathcal{S}_{\textsc{PreTTy}}(\Psi, \Phi_{\partial c}) & \to & \{\tau \in \mathcal{T}y^{\Psi} \mid \underline{\tau} = c\} \\ \alpha & \mapsto & \sigma_c[\alpha] \end{array}$$

is an isomorphism, natural in Ψ. Given a type τ with $\underline{\tau} = c$, we denote the associated substitution

$$\chi_\tau : \Psi \to \Phi_{\partial c} \quad .$$

We have moreover that the maps

$$\begin{array}{rcl} (\mathcal{S}_{\textsc{PreTTy}}/\Phi_{\partial c})\,(\chi_\tau : \Psi \to \Phi_{\partial c}, \xi_c : \Phi_c \to \Phi_{\partial c}) & \to & \mathcal{T}m^{\Psi}_{\tau} \\ \alpha & \mapsto & e(\mathsf{id}_c)[\alpha] \end{array}$$

are also isomorphisms, natural in Ψ. Given a term $t \in \mathcal{T}m^{\Psi}_{\tau}$, we let χ_t denotes the associated substitution over $\Phi_{\partial c}$, in such a way that the following triangle commutes.

$$\begin{tikzcd} \Psi \arrow[r, "\chi_t"] \arrow[dr, "\chi_\tau"'] & \Phi_c \arrow[d, "\xi_c"] \\ & \Phi_{\partial c} \end{tikzcd}$$

Proof. The first point is the reflection of Lemma 3.22 along the equivalence $\mathcal{S}_{\text{PRETTY}} \simeq \hat{\mathcal{C}}^{\text{op}}_f$ of Theorem 3.24. The second is similarly the reflection of the point (i) of Lemma 3.18. □

4.4 Using Lemma 4.3, we may see the interpretation functor $[\![-]\!] : \mathcal{S}_{\text{PRETTY}} \to \hat{\mathcal{C}}^{\text{op}}_f$ as follows. The functor $\Phi_\bullet : \mathcal{C}^{\text{op}} \to \mathcal{S}_{\text{PRETTY}}$ allows us to consider a nerve functor $N : \mathcal{S}_{\text{PRETTY}} \to \hat{\mathcal{C}}^{\text{op}}$ given by $N(\Psi)_c = \mathcal{S}_{\text{PRETTY}}(\Psi, \Phi_c)$. Under the correspondance of Lemma 4.3, this functor coincides with $[\![-]\!]$ on the contexts. Moreover, given a substitution $\alpha : \Theta \to \Psi$, and a term $t \in [\![\Psi]\!]_c$, the definition of χ_t yields $\chi_t \circ \alpha = \chi_{t[\alpha]}$. On the other hand, $[\![\alpha]\!](t) = t[\alpha]$. So N and $[\![-]\!]$ coincides modulo the natural equivalence of Lemma 4.3. Using this interpretation, the equivalence of Theorem 3.24 may be seen as coming from a generalised nerve - realisation adjunction.

Lemma 4.5

Any context $(\Psi, x : \tau)$ where $\underline{\tau} = c$ is obtained as a pullback, as follows:

$$\begin{tikzcd} (\Psi, x : \tau) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Phi_c \arrow[d, "\xi_c"] \\ \Psi \arrow[r, "\chi_\tau"'] & \Phi_{\partial c} \end{tikzcd}$$

Proof. This is a direct consequence of Lemma 4.3 and the comprehension operation property. □

Lemma 4.6

Let $\mathcal{D}$ be a finitely complete category, and $F : \mathcal{C}^{\text{op}} \to \mathcal{D}$ a functor. F extends uniquely to a functor $\tilde{F} : \mathcal{S}_{\text{PRETTY}} \to \mathcal{D}$ which preserves the terminal object and the pullbacks along display maps.

Proof. First of all we show that $\tilde{F}$ is determined on the contexts $\Phi_{\partial c}$ for all c, notice that in the context $\Phi_{\partial c}$, each pair $(x : \tau)$ satisfies $\underline{\tau} \lhd c$. So we may show by induction that $F(\Phi_{\partial c})$ is determined. Indeed, we may construct $\Phi_{\partial c} = \Theta_n$ as a succesion of pullbacks (using Lemma 4.5):

$$\begin{tikzcd} \Theta_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Phi_{b_i} \arrow[d, "\xi_{b_i}"] \\ \Theta_{i-1} \arrow[r, "\chi_{\tau_i}"'] & \Phi_{\partial b_i} \end{tikzcd}$$

Where Θ_k is the context of the first k elements of $\Phi_{\partial c}$, and $(t_i : \tau_i)$ with $\underline{\tau_i} = b_i$ is its k-th element. If we assume $F(\Phi_{\partial b})$ to be known for each $b \lhd c$, then Θ_n must be preserved as a tower of pullbacks along (maps isomorphic to) display maps. That is, $F(\Phi_{\partial c})$ will be caracterised universally as a tower of pullbacks along the $F(\Phi_{b_i}) \to F(\Phi_{\partial b_i})$. In particular, it caracterise the image by F of morphisms whose target is $\Phi_{\partial b_i}$.

Now, we may prove by induction on the length of the context Ψ that $F(\Psi)$ is also determined as a colimit in $\mathcal{D}$, and on the maps whose target is Ψ.

- On the empty context, this is true because F is assumed to send $\varnothing$ to the empty set.
- Let $(\Psi, x : \tau)$ be obtained by ctx-EXT. Then using Lemma 4.5, we have a pullback diagram in $\mathcal{S}_{\text{PRETTY}}$:

$$\begin{tikzcd}
(\Psi, x : \tau) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Phi_c \arrow[d, "\xi_c"] \\
\Psi \arrow[r, "\chi_\tau"'] & \Phi_{\partial c}
\end{tikzcd}$$

And since ξ_c is (isomorphic to) a display map, F must preserve this pullback. So $F((\Psi, x : \tau))$ is defined as a pullback in $\mathcal{D}$, which ensure the desired property.

Now, we may check the functoriality of F. For every context Θ, $F(\Theta)$ has been defined inductively as a pullback of some ξ_c. We will ensure the functoriality of F on maps whose target is Θ by induction on this process.

- Suppose $\Theta = \varnothing$, then $F(\varnothing) = \varnothing$ is terminal so the result is clear.
- Suppose that $\Theta = (\Psi, x : \tau)$, then it a pullback of ξ_c for $c = \underline{\tau}$ along χ_τ, as depicted above. Our inductive hypothesis is that F is functorial on maps whose target are Φ_c or Ψ. A map $\alpha : \Theta' \to (\Psi, x : \tau)$ yields two structural maps $\alpha_1 : \Theta' \to \Phi_c$ and $\alpha_2 : \Theta' \to \Psi$. By definition of F, $F(\alpha)$ is obtained by universal property of $F(\Theta)$, applied to the structural maps $F(\alpha_1)$ and $F(\alpha_2)$. Let $\beta : \Theta'' \to \Theta'$ be a substitution. Then similarly, $\gamma = \alpha \circ \beta$ yield two maps $\gamma_1 : \Theta'' \to \Phi_c$ and $\gamma_2 : \Theta'' \to \Psi$, and $F(\gamma)$ is defined by the universal property of $F(\Theta)$, applied to the maps $F(\gamma_1)$ and $F(\gamma_2)$. By induction hypothesis, $F(\gamma_1) = F(\alpha_1) \circ F(\beta)$ and $F(\gamma_2) = F(\alpha_2) \circ F(\beta)$. Hence, $F(\gamma) = F(\alpha) \circ F(\beta)$ by the universal property of $F(\Theta)$. Hence, F preserves compositions. By definition of F, it also preserves identities.

Finally, we shall check that F preserves the terminal object and the pullbacks along display maps. The first point is by definition of F. As to the second one, consider a pullback along a display map in $\mathcal{S}_{\text{PRETTY}}$, it has the following form:

$$\begin{tikzcd}
(\Psi, x : \tau[\alpha]) \arrow[r, "\langle \alpha \circ \pi', a \mapsto x \rangle"] \arrow[d, "\pi'"'] \arrow[dr, phantom, "\lrcorner", very near start] & (\Phi, a : \tau) \arrow[d, "\pi"] \\
\Psi \arrow[r, "\alpha"'] & \Phi
\end{tikzcd}$$

Lemma 4.5 gives us two other pullback squares, as follows:

$$\begin{tikzcd}
(\Psi, x : \tau[\alpha]) \arrow[r, "\langle \alpha \circ \pi', a \mapsto x \rangle"] \arrow[d, "\pi'"'] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[rr, bend left, "\chi_x"] & (\Phi, a : \tau) \arrow[r, "\chi_a"] \arrow[d, "\pi"] \arrow[dr, phantom, "\lrcorner", very near start] & \Phi_c \arrow[d, "\xi_c"] \\
\Psi \arrow[r, "\alpha"'] \arrow[rr, bend right, "\chi_{\tau[\alpha]}"'] & \Phi \arrow[r, "\chi_\tau"'] & \Phi_{\partial c}
\end{tikzcd}$$

which are the rightmost and the outermost ones. Since both must be preserved by definition of F, the leftmost one must also be preserved, by the pullback pasting lemma. $\square$

Theorem 4.7

There is an equivalence of categories $\mathbf{Mod}(\mathcal{S}_{\text{PRETTY}}) \simeq \hat{\mathcal{C}}$, by restriction along $\mathcal{C}^{\text{op}} \hookrightarrow \mathcal{S}_{\text{PRETTY}}$.

Proof. It is given directly by Lemma 4.6. $\square$

5 Some instances of PRETTY

The Globular Theory $\text{PRETTY}_{\mathbf{G}}$

For this exemple, we consider the reflexive category of globes $\mathbf{G}$. We have the coverings defined in Exemple 1.8, and use the presentation $P([n+1],[n]) = \{\iota_n\}$ With no further relations.
Note that we have $\iota \circ \sigma = \iota \circ \tau = \mathsf{id}$ when those expressions makes sense, so this choice of presentation satisfies the assumptions of 3.1.

5.1 Syntax. We may describe the syntax for terms and types as follows:

$$\begin{array}{rcll} \mathsf{tm} & ::= & x & (x \in \mathcal{A}) \\ & | & \iota^*\mathsf{tm} & \end{array}$$

$$\begin{array}{rcll} \mathsf{tp} & ::= & * & \\ & | & s \to t & (s,t \in \mathsf{tm}) \end{array}$$

We also write more conveniently id_t^k for $\iota^{*k}t$, yielding $\mathsf{tm} = \{\mathsf{id}_x^k\}_{x\in\mathcal{A},k\in\mathbb{N}}$

5.2 Type introduction. The rule TYPE splits into the two following ones:

$$\frac{\Phi\ \mathsf{ctx}}{\Phi\ \vdash * \ \mathsf{type}}\ \text{TYPE-}* \qquad \frac{\Phi\ \mathsf{ctx} \qquad \Phi \vdash s : \tau \qquad \Phi \vdash t : \tau}{\Phi\ \vdash s \to t\ \mathsf{type}}\ \text{TYPE-}\to$$

type introduction rules

5.3 Degeneracies. The rule DEGE boils down to:

$$\frac{\Phi \vdash t : \tau}{\Phi \vdash \mathsf{id}_t : t \to t}\ \text{DEGE}$$

degeneracy rule

5.4 Globular Type Theory. Upon forgeting the degeneracies, one directly recovers the *Globular Type Theory* defined by BENJAMIN, FINSTER and MIMRAM in [2].

The Simplicial Theory $\text{PRETTY}_{\boldsymbol{\Delta}}$

We consider now the category of simplices $\boldsymbol{\Delta}$. We have the coverings defined in Exemple 1.24, and use the presentation $P([n+1],[n]) = \{\sigma_i\}_{0\le i\le n}$ With relations $(\sigma_j \circ \sigma_i)P(\sigma_i \circ \sigma_{j+1})$ when $i \le j$.
We have the identities

$$\sigma_j \circ \delta_i = \begin{cases} \delta_i \circ \sigma_{j-1} & \text{if } i < j \\ \mathsf{id} & \text{if } i \in \{j,\, j+1\} \\ \delta_{i-1} \circ \sigma_j & \text{if } j < i+1 \end{cases}$$

when those expressions makes sense, so this choice of presentation satisfies the assumptions of 3.1.

5.5 Syntax. We may describe the syntax for terms and types as follows:

$$\begin{array}{rcll} \mathsf{tm} & ::= & x & (x \in \mathcal{A}) \\ & | & \sigma_i^*\mathsf{tm} & (i \in \mathbb{N}) \end{array}$$

$$\begin{array}{rcll} \mathsf{tp} & ::= & * & \\ & | & \Delta_{[s_0,\cdots,s_n]} & (n \ge 1,\, s_i \in \mathsf{tm}) \end{array}$$

We write more conveniently $t.i$ for $\sigma_i^* t$, yielding $\mathsf{tm} = \{x.i_1.\cdots.i_n\}_{x\in\mathcal{A},n\in\mathbb{N},i_k\in\mathbb{N}}$. We also write $s \to t$ for the type $\Delta_{[s,t]}$.

5.6 Type introduction. The rule TYPE splits into the three following ones:

$$\frac{\Phi\ \mathsf{ctx}}{\Phi\ \vdash *\ \mathsf{type}}\ \textsc{type-}* \qquad \frac{\Phi\ \mathsf{ctx} \qquad \Phi \vdash s : \tau \qquad \Phi \vdash t : \tau}{\Phi\ \vdash s \to t\ \mathsf{type}}\ \textsc{type-}\to$$

$$\frac{\bigwedge_{0\le i\le n+1} \Phi \vdash t_i : \Delta_{[s^i_0,\cdots,s^i_n]} \qquad \bigwedge_{0\le i<j\le n+1} s^i_{j-1} = s^j_i}{\Phi\ \vdash \Delta_{[t_0,\cdots,t_n]}\ \mathsf{type}}\ \textsc{type-}\Delta \quad (n\ge 2)$$

type introduction rules

5.7 Degeneracies. The rule DEGE splits into the following ones:

$$\frac{\Phi \vdash t : *}{\Phi \vdash t.0 : t \to t}\ \textsc{dege-}* \qquad \frac{\Phi \vdash t : \Delta_{[s_0,\cdots,s_n]}}{\Phi \vdash t.k : \Delta_{[s_0.(k-1),\cdots,s_{k-1}.(k-1),t,t,s_{k+1}.k,\cdots,s_n.k]}}\ \textsc{dege-}\Delta \quad (0\le k\le n)$$

degeneracy rules

The equality $s \equiv t$ of two terms may be decided quickly by using the following argument:

Lemma 5.8 : Normal form of terms

Each term $t \in \mathsf{tm}$ may be written as $x.i_1.\cdots.i_n$ for a unique x, n and i_k's such that $i_1 \le \cdots \le i_n$.

The Cubical Theory $\mathrm{PRETTY}_\square$

We finally consider the exemple of the category of cubes $\square$. We have the coverings defined in Exemple 1.25, and use the presentation $P([n+1],[n]) = \{\sigma_i\}_{0\le i\le n}$ With relations $(\sigma_j \circ \sigma_i)P(\sigma_i \circ \sigma_{j+1})$ when $i \le j$.
We have the identities

$$\sigma_j \circ \delta_{i,\varepsilon} = \begin{cases} \delta_{i,\varepsilon} \circ \sigma_{j-1} & \text{if } i<j \\ \mathsf{id} & \text{if } i=j \\ \delta_{i-1,\varepsilon} \circ \sigma_j & \text{if } i>j \end{cases}$$

when those expressions makes sense, so this choice of presentation satisfies the assumptions of 3.1.

5.9 Syntax. We may describe the syntax for terms and types as follows:

$$\begin{array}{lcll} \mathsf{tm} & ::= & x & (x \in \mathcal{A}) \\ & | & \sigma_i^* \mathsf{tm} & (i \in \mathbb{N}) \end{array}$$

$$\mathsf{tp} \quad ::= \quad \square_{[(s^0_1,s^1_1),,\cdots,(s^0_n,s^1_n)]} \quad (n\ge 0,\ s^\varepsilon_i \in \mathsf{tm})$$

We write more conveniently $t.i$ for $\sigma_i^* t$, yielding $\mathsf{tm} = \{x.i_1.\cdots.i_n\}_{x\in\mathcal{A}, n\in\mathbb{N}, i_k\in\mathbb{N}}$.
We also write $*$ for the type $\square_{[]}$.

5.10 Type introduction. The rule TYPE splits into the two following ones:

$$\frac{\bigwedge_{\substack{1\le i\le n \\ 0\le\varepsilon\le 1}} \Phi \vdash t^\varepsilon_i : \square_{[(s^{\varepsilon,i,0}_1, s^{\varepsilon,i,1}_1),\cdots,(s^{\varepsilon,i,0}_n, s^{\varepsilon,i,1}_n)]} \qquad \bigwedge_{\substack{1\le i<j\le n \\ 0\le \varepsilon,\eta\le 1}} s^{\varepsilon,i,\eta}_{j-1} = s^{\eta,j,\varepsilon}_i}{\Phi\ \vdash \square_{[(s^0_1,s^1_1),\cdots,(t^0_n,t^1_n)]}\ \mathsf{type}}\ \textsc{type-}\square \quad (n\ge 0)$$

type introduction rules

5.11 Degeneracies. The rule DEGE may be written as follows:

$$\frac{\Phi \vdash t : \square_{[(s_1^0, s_1^1), \cdots, (s_n^0, s_n^1)]}}{\Phi \vdash t.k : \square_{[(s_1^0.(k-1), s_1^1.(k-1)), \cdots, (s_{k-1}^0.(k-1), s_{k-1}^1.(k-1)), (t,t), (s_{k+1}^0.k, s_{k+1}^1.k), \cdots, (s_n^0, s_n^1).k]}} \text{ DEGE} \quad (1 \leq k \leq n)$$

degeneracy rules

The equality $s \equiv t$ of two terms may be decided quickly by using the following argument:

Lemma 5.12 : Normal form of terms

Each term $t \in \mathrm{tm}$ may be written as $x.i_1.\cdots.i_n$ for a unique x, n and i_k's such that $i_1 \leq \cdots \leq i_n$.

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