(∞, ω)-Categories in Spatial Type Theory

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Abstract

Using the Spatial Type Theory introduced by M. SHULMAN in [29], we present a type theory modeled by the ∞-topos of presheaves over the category Θ. In particular, we may carve out a type of weak (∞, ω)-categories by defining suitable SEGAL and completeness conditions, as defined by C. REZK in [24]. In many regards the approach we have taken follows the ideas introduced by E. RIEHL and M. SHULMAN in [25]. In this paper we lay down those definitions and prove some very simple properties, as a proof of concept for further development.

Contents

1 Introduction 2

2 Main Ideas 3

3 The Category Θ 4

4 The Type Theory 6

5 (∞, ω)-Categories 11

6 Some Results 14

7 The Subuniverse of Codiscrete Types 21

8 (∞, n)-Categories 23

9 Directed Homotopy 25

10 Perspectives and Conclusion 27

A Semantics of CellTT 30

1 Introduction

1.1 Homotopy and type theory. In logic and type theory, one of the major advances of the last two decades has been the establishment of a so-called "homotopy" type theory (abbreviated HoTT), whose types are naturally interpreted as spaces up to homotopy, that is ∞-groupoids [30]. These objects have been conceived and studied in algebraic topology for over half a century, [15, 16, 23], and still resist simple axiomatic description. The advantage of such a theory is that it allows an external and synthetic approach to these spaces up to homotopy: we have a language whose objects behave like ∞-groupoids [31], we can talk about them and establish theorems on them, without ever having to define them precisely. Another inherent advantage of type theory is its computational content. Mathematical functions are replaced by programs that systematically calculate. We can also use computers to ensure that our reasoning is correct (this is the idea behind proof assistants as ROCQ, AGDA, LEAN *etc.*).

Finally, another major interest of this new theory, anticipated by VOEVODSKY in the 2000s [32], was its application to the formalization of mathematics in general. Indeed, it is for example possible to think of a set as a discrete space, whose homotopy is concentrated in degree 0. Or of a logical proposition as a space whose homotopy is trivial in a suitable sense. Generally speaking, certain recent developments in mathematics or theoretical physics have shown the importance of so-called "higher" structures, or even "up to homotopy." Such as the n-categories of cobordisms in field theory [3, 4], or the stacks used in algebraic geometry [10]. For instance, it is possible to formalize naturally homotopic algebra in HoTT, such as ∞-groups [7].

Today, HoTT has become a language of mathematical discussion. Similar to set theory, but distinguished by its constructive flavor and its ability to formulate synthetically theorems from homotopy theory.

1.2 A Directed Homotopy Type Theory. More generally than ∞-groupoids, we would like to study (∞, ω)-categories, which are weak categories whose morphisms do not necessarily have an inverse. On the one hand, these structures appear very naturally in mathematics (for example, ∞-groupoids form a ∞-category), and we would also like to be able to study them in a synthetic and mechanized way. On the other hand, recent developments in computer science related to the study of the semantics of concurrent programs have given rise to the notion of *directed spaces* [9], which naturally give rise to ∞-categories. Indeed, directed paths in these spaces correspond to program executions that respect the direction of time.

Furthermore, morphisms correspond to equalities, and the theory of rewriting in computer science has shown that it is often relevant to orient these equalities (if two elements represent the same class, one can often find one that is "better" according to certain criteria), in order to access new computational tools and calculate invariants or demonstrate consistency properties.

Finally, while ∞-groupoids are complicated to manipulate, we know of models that remain combinatorially accessible (for example, simplicial models based on KAN complexes). The currently known definitions of higher categories are extremely difficult to manipulate; thus, the introduction of new computational tools to master their combinatorics and verify the associated constructions would be very useful.

While the idea of a directed variant of type theory has been present almost since the beginning of homotopy type theory, current proposals are not entirely satisfactory. The most advanced proposal is that of RIEHL and SHULMAN [25]. Based on an extension of MARTIN-LÖF type theory (MLTT), inspired by cubical models of homotopy type theory, and guided by the model of SEGAL-spaces. It has allowed the formalization of non-trivial results but is limited to 1-categories (all n-morphisms of which are invertible for $n \geq 2$). Another approach has been proposed by NORTH [20] which in some sense is orthogonal to the direction explored by Simplicial Type Theory and the work presented here.

1.3 Differences with the $(\infty, 1)$-case. In Simplicial Type Theory, hom types are discrete, in the sense that they are ∞-groupoids. This occurs because Rezk-types are "only" $(\infty, 1)$-categories. While in the (∞, ω)-setting, hom types of (∞, ω)-categories should remain (in general, non-discrete) (∞, ω)-categories. This crucial difference allows the authors of STT to define hom types as usual mapping spaces

$$\mathsf{hom}_A(x, y) = \sum_{f:I \to A} f(0) = x \times f(1) = y$$

where I - a directed interval - is interpreted as the YONEDA embedding of the 1-simplex (seen as an object of Δ, in the $(\infty, 1)$-case, and as an object of Θ, in the (∞, ω)-case). Howether, when working with (∞, ω)-categories, this mapping type behaves differently to the hom type. A 2-cell in A should be a map $I \to (I \to A)$ with boundary conditions, which by Curryfication, should be the same as a map $I^2 \to A$ with boundary conditions. Whereas I^2 is only a 1-categorical object. Hence, we must find a way to define the hom type differently.

1.4 Introducing $\flat$. One way to do so is to work with an idempotent modality which is comonadic, for instance building upon *Crisp Type Theory*, as introduced by M. SHULMAN in [29]. Here, our flat modality comes from an adjunction :

$$\mathcal{S} \underset{\mathsf{ev}_*=(F\mapsto F(*))}{\overset{\delta}{\underset{\perp}{\rightleftarrows}}} [\Theta^{\mathrm{op}}, \mathcal{S}]$$

where $\mathcal{S}$ is the $(\infty, 1)$-categeory of spaces, and $\delta(X)$ is the constant functor equal to X. We then have an idempotent comonad $\flat = \delta \circ \mathsf{ev}_* : [\Theta^{\mathrm{op}}, \mathcal{S}] \to [\Theta^{\mathrm{op}}, \mathcal{S}]$. This setting falls in the broader one of local toposes which should be model of the *spatial* fragment of cohesive HoTT according to M. SHULMAN (see Remark 1.2 in [29]).

This modality allows us to speak about the types of "points" X_* of a type X by seing it as a discrete (i.e. constant) presheaf (i.e. type). When working with a type which is an (∞, ω)-category, it should computes its core.

1.5 Spatial Type Theory. Recently, M. SHULMAN has introduced a type theory extending Homotopy Type Theory by adjoining two adjunct modalities $\flat \dashv \sharp$ [29]. This setting gives a language suitable for talking about any local ∞-topos. For an exposition of this notion in the case of 1-categories, one may refer to [13]. The ∞-categorical notion may be found at [1].

2 Main Ideas

We start by presenting the outline of the paper and give an overview of the main ideas and constructions that will be introduced in the following sections.

2.1 The category Θ. In Section 3, we start by presenting the category of pasting schemes Θ, which may be thought as a full subcategory of strict ω-categories. This category will play a central role in our work, as it is the category of shapes on which our model of (∞, ω)-categories is built. Then, we also give a combinatorial description of the objects and morphisms of Θ, which will be usefull for formalizing it in Section 4.

2.2 The type theory. We then describe our type theory in Section 4. Informally, this will be an extension of HoTT with an idempotent comodality $\flat$ and a right adjoint $\sharp \vdash \flat$ (which form the *spatial fragment* of cohesive homotopy type theory [29]), Together with seven postulates. It is intended to have a semantic in the model category $\mathsf{sPsh}(\Theta)$, that is, in the higher category of presheaves over Θ, in which one may define (∞, ω)-categories. So our main idea is to assume every type to look like a presheaf over an internalization of the category Θ.

We sketch the seven postulates we will consider:

- *(YONEDA embedding)* There is a wild functor $\text{よ} : \Theta \to \mathcal{U}$, where Θ is the aforementioned internalization of the category Θ in type theory. This functor is assumed to be crisply fully faithfull, and we think of it as a YONEDA embedding.
- *(cellular cohesion)* There is a notion of $\flat$-discreteness for a type (*i.e.* $\flat$-modal types). And there is a notion of cellular discreteness for types, which mean this type have all its cells degenerated. We postulate the two notions to coincide. This axioms fits in the broader setting of punctual cohesion defined in section 8 of [29].
- *(equivalences are objectwise)* Using our YONEDA embedding, each crisp type A now comes with a (discrete) type of P-cells A_P for each P, and every map $f : A \to B$ induces maps $f_P : A_P \to B_P$ on P-cells. We ask for f to be an equivalence whenever all the f_P are.

- *(suspension)* We postulate a pushout preserving wild functor $\$' : \mathcal{U} \to \mathcal{U}_{\bullet\bullet}$ from types to bipointed types, which we call the *suspension*. It should be thought as a directed version of the suspension operation introduced in [30].

- *(hom types)* We postulate the existence of hom-types as a right adjoint to the suspension operation previously postulated.

- *(connectedness of representables)* We assume the $(-)_P : \mathcal{U} \to \mathcal{U}$ operation to preserves pushouts. This is motivated by the objectwiseness of colimits in presheaves categories, or by connectedness of representables in a presheaf category.

- *(coverage)* We assume that every element $x : A$ in a crisp type $A : \mathcal{U}$ lie in the image of a P-cell $c : \flat(\text{よ}(P) \to A)$ for some P.

2.3 (∞, ω)-categories. In Section 5 we will define the key notions of Segalness and Completeness of a type, which will yield our definition of (internal) (∞, ω)-categories. That is, we will introduce a proposition which will carve the type of higher categories as a subtype of the universe $\mathcal{U}$. It will be given by those types which are both SEGAL and complete, in a sense generalizing that introduced in [25].

2.4 Some results. In Section 6, we will present a selection of usefull facts about higher categories as defined in Section 5 and more generally about the type theory introduced in Section 4. For instance, it will be shown that the homotopy level may be computed objectwise, or that infinity categories are preserved by sums and pullbacks.

2.5 The subuniverse of codiscrete types. In Section 7, we explain how the $\sharp$ modality of Spatial Type Theory (the right adjoint to $\flat$) behaves in our type theory, and gives a caracterization of crisp $\sharp$-modal types. Namely, they will be those types which see the representables as their core ∞-groupoid. In particular it implies that all their hom-types are contractibles.

2.6 (∞, n)-category. We move on to a quick description of (∞, n)-categories for any $0 \leq n \leq \infty$ in Section 8. We will give several caracterizations of this notions, together with basic facts about them as a sanity check.

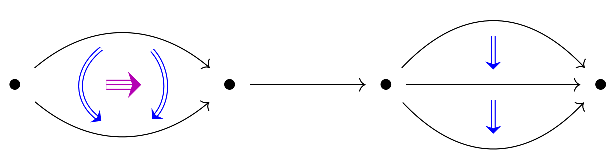
2.7 Directed homotopy. We sketch in Section 9 some basics of directed homotopy theory that one could develop in our formalism. We give a definition of a reduced directed suspension $\overrightarrow{\Sigma}$ as an example of a *localization type*, and propose an inductive definition of directed n-spheres. As a proof of concept for further development, we show that there is a directed counterpart $\overrightarrow{\Omega} \dashv \overrightarrow{\Sigma}$ to the usual loopspace-suspension adjunction.

2.8 Semantics. As a motivation for the work we have provided in this article, we sketch a semantic of our type theory in the ∞-topos of presheaves over Θ as an appendix. It is not intended to be a fully precise description, but rather an informal justification for our postulates.

3 The Category Θ

3.1 The category Θ (or *cell*-category) has been first introduced by A. JOYAL in [14] as a shape category suitable for defining weak (∞, ω)-categories. It was then used and studied by various authors for this same purpose, and especially showing properties of its category of presheaves [6, 8].

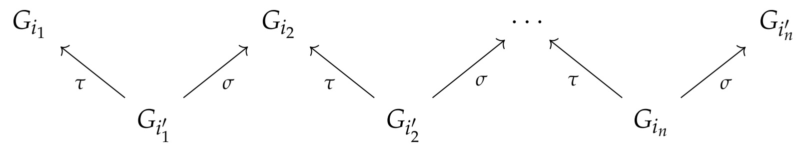
3.2 Pasting schemes. Pasting schemes constitute the objects of the category Θ. They consist of pastings of formal globes, as depicted below:



Formally, let $\mathbb{G}$ denotes the category of formal globes:

$$G_0 \underset{\tau_0}{\overset{\sigma_0}{\rightrightarrows}} G_1 \underset{\tau_1}{\overset{\sigma_1}{\rightrightarrows}} G_2 \underset{\tau_2}{\overset{\sigma_2}{\rightrightarrows}} \cdots$$

whose objects are the $(G_i)_{i\in\mathbb{N}}$ and morphisms the $\sigma_i, \tau_i : G_i \to G_{i+1}$ subjects to the relations $\sigma \circ \sigma = \tau \circ \sigma$ and $\sigma \circ \tau = \tau \circ \tau$. Then the pasting schemes are the diagrams of the following shape in $\mathbb{G}$:



where we have denoted $\sigma = \sigma \circ \cdots \circ \sigma$ and $\tau = \tau \circ \cdots \circ \tau$, ommiting the indices for readability. Hence, pasting schemes are also called *globular sums* and may be intuitively thought as the colimits of those diagrams in the category of globular sets $\hat{\mathbb{G}}$. For a further account of thoses objects and their application to higher categories, one should check the work D. ARA [2]. In fact, any such diagram will induce a unique strict ω-category, and this embedding allow us to define their morphisms as the morphisms of the strict ω-categories they generate. See the work of C. BERGER [5] for more details on this approach.

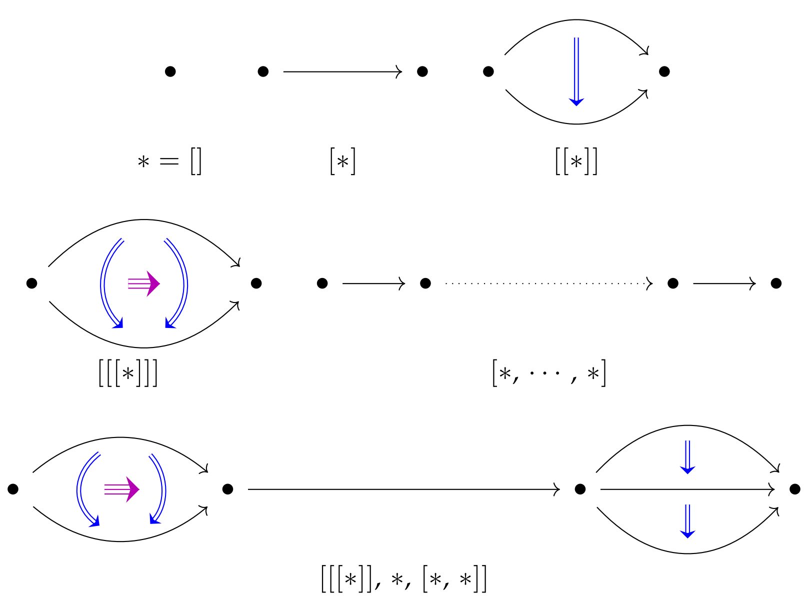
The category Θ, combinatorially

For our purpose, we will use a more direct and combinatorial description of the category Θ, suitable for a type-theoretic reformulation.

Definition 3.3 : Pasting schemes

The pasting schemes are the objects of the category Θ, they are inductively defined as follows:
A pasting scheme P is a list of pasting schemes $P = [P_1, \cdots, P_n]$.

3.4 As this very concise definition may be a bit puzzling, we provide some pictures of pasting schemes, labelled by their name according to Definition 3.3.

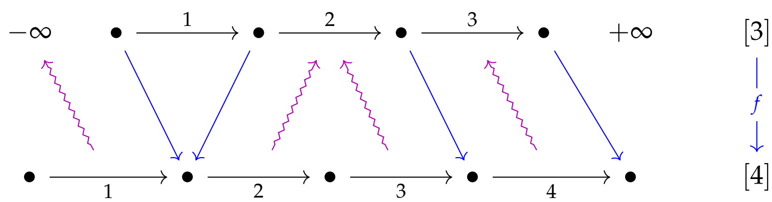


Using this description, one should think of the pasting scheme $P = [P_1, \cdots, P_n]$ as the glueing of the suspensions of the pasting schemes $P_1, \cdots, P_n$ along their endpoints.

3.5 We may now define combinatorially the maps between two pasting schemes. We first introduce the notion of *dual map*. We use the standard notation $[n] = \{0, \cdots, n\}$ ($[n-1] = \emptyset$) equiped with the linear order $0 \leq 1 \leq \cdots \leq n$. We also denote $\langle n \rangle = \{1, \cdots, n\}$ and $\langle n \rangle_{++} = \{-\infty\} + \langle n \rangle + \{+\infty\}$ with the intuitive linear order.

> **Definition 3.6 : Dual map**
>
> Let $f : [n] \to [m]$ be a non-decreasing map. Its *dual* map $f^\vee : \langle m \rangle \to \langle n \rangle_{++}$ is defined by
>
> $$f^\vee(k) = \begin{cases} -\infty & \text{if } f(0) \geq k \\ i & \text{if } f(i-1) < k \leq f(i) \\ +\infty & \text{if } f(n) < k \end{cases}$$
>
> Note that $f^\vee$ extend canonically as a map $f^\vee : \langle m \rangle_{++} \to \langle n \rangle_{++}$, and as such $(-)^\vee$ is functorial.

The following sketch should yield intuition for this construction:

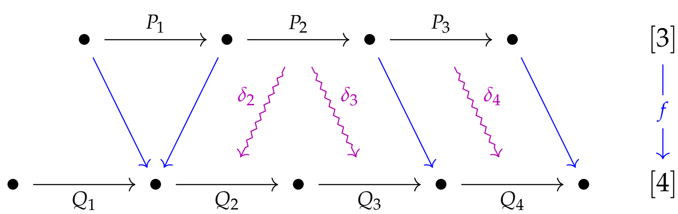


where we we have depicted the map f in blue and its dual map $f^\vee$ in squiggly purple arrows.

We now give the inductive definition of morphisms between pasting schemes.

> **Definition 3.7 : Morphism of pasting schemes**
>
> Let $P = [P_1, \cdots, P_n]$ and $Q = [Q_1, \cdots, Q_m]$ be two pasting schemes. A *morphism* $P \to Q$ between them is the following data.
>
> - A non-decreasing map $f : [n] \to [m]$.
> - For each $k \in \langle m \rangle$ such that $f^\vee(k) \in \langle n \rangle$, a morphism $\delta_k : P_{f^\vee(k)} \to Q_k$.
>
> We pack this data as a pair (f, δ).

This situation is depicted below:



Where the purple squiggly arrows are now morphisms of pasting schemes.

3.8 Composition and identities. The composition of two morphisms (f, δ) and (g, γ) is defined as $(g \circ f, \gamma \circ \delta)$ where $(\gamma \circ \delta)_k := \gamma_k \circ \delta_{g^\vee(k)}$. The identity morphism $[P_1, \cdots, P_n] \to [P_1, \cdots, P_n]$ is defined inductively as $\left(\mathrm{id}_{[n]}, (\mathrm{id}_{P_i})_{i \in \langle n \rangle}\right)$.

4 The Type Theory

The base type system we are considering is Crisp Type Theory, with the $\flat$-modality introduced by M. Shulman in [29]. We introduce the notation $\prod_{x::X} \Phi(x)$ (resp. $\sum_{x::X} \Phi(x)$) for $\prod_{x:X} \Phi(x_\flat)$ (resp. $\sum_{x:X} \Phi(x_\flat)$). We then add axioms which holds in the Θ-spaces model.

The type of pasting schemes

We first define the type of pasting schemes and expose some basic constructions involving it.

Definition 4.1 : Pasting schemes, suspension

There is an inductive type PS of pasting schemes whose sole constructor is:

$$\mathsf{cons} : \mathsf{PS}\ \mathsf{List} \to \mathsf{PS}$$

We write more concisely $[P_1, \cdots, P_n]$ for the type $\mathsf{cons}\ [P_1, \cdots, P_n]$. We let \$ denote the *suspension* operation

$$\begin{aligned} \$: \mathsf{PS} &\longrightarrow \mathsf{PS} \\ P &\longmapsto [P] \end{aligned}$$

Definition 4.2 : Morphisms $P \to_{\mathsf{PS}} Q$

Let P, Q be two pasting schemes, there is a type of morphisms $P \to_{\mathsf{PS}} Q$ between them, which is also defined inductively, following Definition 3.7. its objects are dependent pairs $(f, \delta) : P \to_{\mathsf{PS}} Q$, as in the definition *loc.cit.*
Following paragraph 3.8, there is a composition and identities:

$$- \circ - : \prod_{P, Q, R\, :\, \mathsf{PS}} (Q \to_{\mathsf{PS}} R) \to (P \to_{\mathsf{PS}} Q) \to (P \to_{\mathsf{PS}} R)$$

$$\mathsf{id} : \prod_{P\, :\, \mathsf{PS}} P \to_{\mathsf{PS}} P$$

Theorem 4.3 : The category Θ

The types PS and $P \to_{\mathsf{PS}} Q$ for $P, Q : \mathsf{PS}$ have a decidable equality, so both are sets according to HEDBERG's theorem (see thm. 7.2.5 in [30]).
Moreover, pasting schemes and their morphisms with $\circ$ and id form a category in the sense of Definition 9.1.6 of the HoTT book [30].

4.4 Some notations. We let $* = []$ denote the *terminal* pasting scheme (it is!). Then $[n]$ denotes the list $[*, *, \cdots, *]$ of length n, and $\mathcal{O}_n = \$^n *$ is the list $[[\cdots[]\cdots]]$ containing $n+1$ pairs of brackets.
Remark also that any pasting scheme $P = [P_1, \cdots, P_n]$ comes equiped with a left and right endpoints

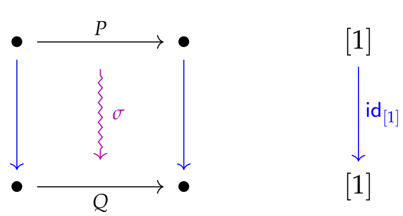
$$\mathsf{left}_P = (0, \_) : * \to_{PS} P \qquad \mathsf{right}_P = (n, \_) : * \to_{PS} P$$

Definition 4.5 : Suspension of morphisms

For $\sigma = (f, \delta) : P \to_{\mathsf{PS}} Q$ a morphism of pasting schemes, there is a *suspended* morphism

$$\$\sigma :\equiv (\mathsf{id}_{[1]}, (\sigma))$$

This construction is functorial, and any suspended map preserves the endpoints. As such we say it is *bipointed*.



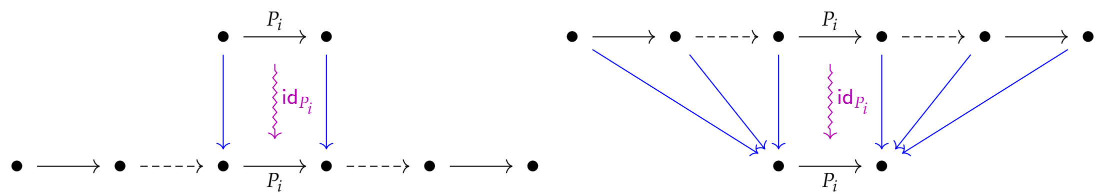
Definition 4.6 : Inclusion and retraction morphisms

Let $P = [P_1, \cdots, P_n]$ be a pasting scheme and $1 \leq i \leq n$. There are *inclusion* $\subseteq_i : \$P_i \to_{\mathsf{PS}} P$ and *retraction* $\pi_i : P \to_{\mathsf{PS}} \$P_i$ morphisms defined as

$$\subseteq_i :\equiv \left(x \mapsto i + x,\ (\mathsf{id}_{P_i})\right) \qquad \pi_i :\equiv \left(x \mapsto \begin{cases} i & \text{if } x \leq i \\ i+1 & \text{if } x > i \end{cases},\ (\mathsf{id}_{P_i})\right)$$

One may check that the retraction π_i is indeed a retraction of the inclusion $\subseteq_i$.

Here are illustrations of the inclusion morphism $\subseteq_i$ and the retraction morphism π_i:



Definition 4.7 : Dimension

The dimension $\mathsf{dim}(P)$ of a pasting scheme $P : \mathsf{PS}$ is defined inductively as follows.

- When $P = *$ is the empty list: $\mathsf{dim}(P) = 0$.
- When $P = [P_1, \cdots, P_n]$ with $n > 0$, $\mathsf{dim}(P) = \max\{\mathsf{dim}(P_i)\}_{1 \leq i \leq n}$.

Definition 4.8 : Source and target morphisms

We define inductively on $P : \mathsf{PS}$ and $k \in \mathbb{N}$, its k-boundary $\partial^k P : \mathsf{PS}$, with two morphisms $\mathsf{src}_P^k, \mathsf{tgt}_P^k : \partial^k P \to_{\mathsf{PS}} P$.

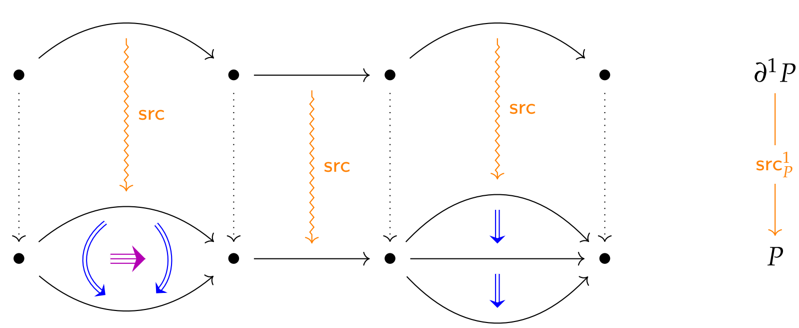
- When $k = 0$, $\partial^0 P :\equiv *$, with $\mathsf{src}_P^0 :\equiv \mathsf{left}_P$ and $\mathsf{tgt}_P^0 :\equiv \mathsf{right}_P$.
- When $P = [P_1, \cdots, P_n]$ with $k > 0$, we let

$$\partial^k P :\equiv [\partial^{k-1} P_1, \cdots, \partial^{k-1} P_n]$$

$$\mathsf{src}_P^k :\equiv \left(\mathsf{id}_{[n]},\ (\mathsf{src}_{P_i}^{k-1})_i\right) \quad \text{and} \quad \mathsf{tgt}_P^k :\equiv \left(\mathsf{id}_{[n]},\ (\mathsf{tgt}_{P_i}^{k-1})_i\right) \quad .$$

When $\mathsf{dim}(P) > 0$, we denote ∂P (resp. src_P, tgt_P) the pasting scheme $\partial^{\mathsf{dim}(P)-1}$ (resp. the morphisms $\mathsf{src}_P^{\mathsf{dim}(P)-1}$, $\mathsf{tgt}_P^{\mathsf{dim}(P)-1}$).

4.9 This construction is illustrated below.



The axioms of CellTT

We now introduce the axioms we need to reproduce constructions from higher category theory.

Postulate 1 : Yoneda embedding

For each pasting scheme P, we postulate a fully-faithful *wild functor* $よ : \Theta \to \mathcal{U}$. That is:

- For each P : PS, a type $よ(P) : \mathcal{U}$.
- For each $\sigma : P \to_{\mathsf{PS}} Q$, a function $よ(\sigma) : よ(P) \to よ(Q)$.
- For each P : PS, an identity $よ(\mathsf{id}_P) = \mathsf{id}_{よ(P)}$.
- For each composable morphisms $\sigma : P \to_{\mathsf{PS}} Q$ and $\tau : Q \to_{\mathsf{PS}} R$, an identity $よ(\tau \circ \sigma) = よ(\tau) \circ よ(\sigma)$.
- Such that for all P, Q : PS, the map $\flat よ : \flat(P \to_{\mathsf{PS}} Q) \to \flat(よ(P) \to よ(Q))$ is an equivalence.

We also refer to $よ(P)$ as the *fibrant realization* of P, as opposed to the *non-fibrant* or *cellular* realization that will be introduced later.

4.10 We let $I :\equiv よ(\Delta^1) \equiv よ(\mathcal{O}_1)$ and $\mathsf{D}_n :\equiv よ(\mathcal{O}_n)$. Recall from [29] that a type A is called $\flat$-discrete iff the counit $\flat A \to A$ is an equivalence (which boils down to asking it to admit a section)

Postulate 2 : Celular cohesion

A type A is said to be *cellularly-discrete* iff for all P : PS, the canonical map $A \to (よ(P) \to A)$ is an equivalence. If $A :: \mathcal{U}$ is a crisp type, we postulate both notions of discreteness to be the same. That is:

$$\text{is-}\flat\text{-discrete}(A) \leftrightarrow \text{is-cellularly-discrete}(A)$$

From now how, we refer to both notions as *discreteness*, and write is-discrete(A) for this property.

4.11 For any crisp type $A :: \mathcal{U}$, we let $A_P :\equiv \flat(よ(P) \to A)$ and $A_n :\equiv A_{\mathcal{O}_n}$ when $n : \mathbb{N}$. Any map $f : A \to B$ induces a map $f_P : A_P \to B_P$. Then we assume the following.

Postulate 3 : Equivalences are objectwise

A crisp map $f :: A \to B$ is an equivalence iff all the f_P are. That is:

$$\left(\prod_{P::\mathsf{PS}} \text{is-equiv}(f_P) \right) \to \text{is-equiv}(f) \quad .$$

4.12 As a corollary of Postulate 3, any object is contractible iff it is objectwise. Especially, $よ(*) \simeq \mathbb{1}$ is contractible. This implies that any fibrant realisation $よ(P)$ is equiped with a left and a right endpoint

$$\mathsf{left} :\equiv よ(\mathsf{left}) : \mathbb{1} \simeq よ(*) \to よ(P) \quad \text{and} \quad \mathsf{right} :\equiv よ(\mathsf{right}) : \mathbb{1} \simeq よ(*) \to よ(P) \quad .$$

Moreover, this makes our Postulate 2 fit into the setting of *punctual cohesion* as introduced in paragraph 8.2 of [29]. Which in turn makes PS and $P \to_{\mathsf{PS}} Q$ discrete types according to Theorem 4.3 and the opposite implication of Lemma 8.15 in *loc.cit.* (which does not require LEM to hold).

4.13 Bipointed types and maps. In the following, we write $\mathcal{U}_{\bullet\bullet}$ for the type of *bipointed types* $\sum_{X : \mathcal{U}} X \times X$ and $X \to_{\bullet\bullet} Y$ for the type of bipointed maps (preserving both endpoints) between bipointed types X and Y.

Postulate 4 : Suspension

We assume:

- A pushout-preserving [a] (crisp) wild functor $\$' :: \mathcal{U} \to \mathcal{U}_{\bullet\bullet}$.
- For all P :: PS, a (crisp) *intertwining* map $\beta_P :: よ(\$P) \to_{\bullet\bullet} \$'(よP)$.

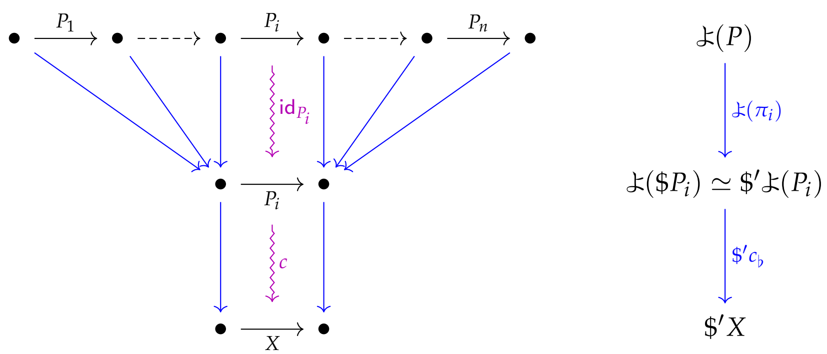
Such that the canonical map $\mathbb{1} + \sum_i X_{P_i} + \mathbb{1} \to (\$'X)_P$ is an equivalence. We call $\$'$ the suspension functor, as it extends $\$$ to all types.

[a] We could also assume 2-coherence to ensure that $\$'$, as a left adjoint to hom (see Postulate 5) preserves colimits (see [22]).

4.14 We unfold Postulate 4 and define precisely what is the "canonical map" mentionned above. On the first summand, the map $\mathbb{1} + \sum_i X_{P_i} + \mathbb{1}$ picks the constant map $\text{よ}(P) \to \$'X$ equals to the left point of $\$'X$. On the last summand, it picks the map constant equal to the right point of $\$'X$. On the middle summand, a cell $c : \flat(\text{よ}(P_i) \to X)$ is sent to the map

$$\left(\text{よ}(\pi_i) \circ \beta_{P_i} \circ \$'(c_\flat)\right)^\flat$$

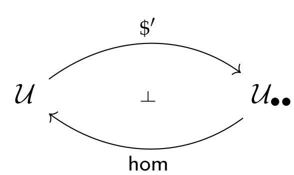
which is depicted with the following illustration:



In particular, when instancied at $P' = \$P$, the postulate implies that the intertwining map β_P itself is an equivalence.

Postulate 5 : hom types

We assume a crisp 1-coherent right adjoint hom to the suspension functor $\$' : \mathcal{U} \to \mathcal{U}_{\bullet\bullet}$.



We denote $\mathsf{hom}_A(x, y)$ its value at $(A, x, y) :: \mathcal{U}_{\bullet\bullet}$.

Remark 4.15
What we really mean by adjunction in Postulate 5 is that we have a (1-coherent) natural equivalence

$$\flat(A \to \mathsf{hom}_B(x, y)) \cong \flat(\$'A \to_{\bullet\bullet} (B, x, y))$$

for any crisp $A :: \mathcal{U}$ and crisp bipointed type $(B, x, y) :: \mathcal{U}_{\bullet\bullet}$. Thus, this should be thought as a usual ∞-categorical adjunction, while asking for an equivalence

$$(A \to \mathsf{hom}_B(x, y)) \cong (\$'A \to_{\bullet\bullet} (B, x, y))$$

would yield an enriched adjunction (enriched in presheaves over Θ), which would be wrong semantically. We denote the unit of this adjunction $\mathsf{merid} : A \to \mathsf{hom}_{\$'A}(\mathsf{left}, \mathsf{right})$. For two points $x, y : \flat B$, we will also write $\mathsf{hom}_B(x, y)$ for $\mathsf{hom}_B(x_\flat, y_\flat)$, thus viewing hom_B as a function

$$\mathsf{hom}_B : \flat B \times \flat B \to \mathcal{U} \quad .$$

Postulate 6 : Connectedness of representables

For any P : PS, the functor $(-)_P$ is required to preserve sums.

Postulate 7 : Coverage

For a crisp type $A :: \mathcal{U}$, the map

$$\sum_{P::PS} \sum_{c:A_P} \text{よ}P \to A$$

is an effective epimorphisme (i.e. its fibers are inhabited.)

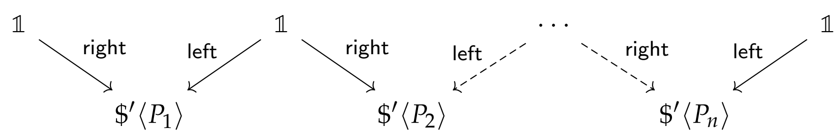
4.16 This last axiom is motivated by Corollary 5.1.6.11 from the work of J. LURIE [19]. Roughly, it is a way to enforce every type to be a colimit of representables (of the form $\text{よ}(P)$).

5 (∞, ω)-Categories

5.1 We now expose some constructions we are able to make inside the type theory we have defined in Section 4. We first introduce the notion of *cellular* (or *non-fibrant*) *realization* of a pasting scheme. Then we will be able to formulate the definition of SEGAL-type, the completeness condition and finally (∞, ω)-categories. In many regards, our work follows the ideas gathered in F. LOUBATON's thesis [17].

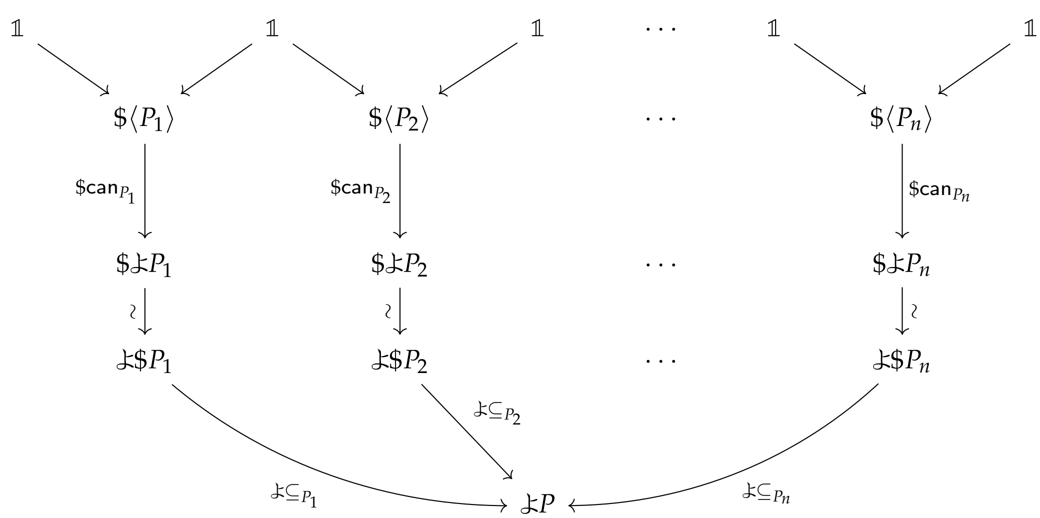
Definition 5.2 : Cellular realization

Let $P = [P_1, \cdots, P_n]$: PS be a pasting scheme. We define inductively its *cellular realization* $\langle P \rangle$ as the colimt of the following diagram:



Note that assuming the existence of pushouts in our type theory (for instance, using a Higher Inductive Type) suffices to compute iteratively this colimit.

5.3 Canonical map $\langle P \rangle \to \text{よ}(P)$. We may define inductively a canonical map $\mathsf{can}_P : \langle P \rangle \to \text{よ}(P)$. Indeed, suppose $P = [P_1, \cdots, P_n]$ and having already defined maps can_{P_i} for all P_i's. Then we may form the following cocone:



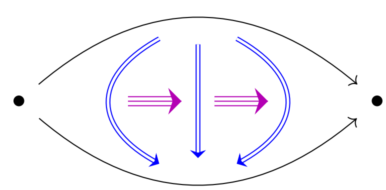
Thus, by the universal property of $\langle P \rangle$, it factors as the desired canonical map $\mathsf{can}_P : \langle P \rangle \to \text{よ}(P)$.

> **Definition 5.4 : SEGAL-type**
>
> A crisp type $A :: \mathcal{U}$ is said to be a SEGAL-type whenever for all $P :: \mathsf{PS}$, the canonical map yield an equivalence
> $$\flat(よ(P) \to A) \xrightarrow{\sim} \flat(\langle P \rangle \to A) \quad .$$
> We write is-SEGAL(A) the associated proposition.

Exemple 5.5
We will see later on that any discrete type (which should be thought as a space, or an ∞-groupoid) is a SEGAL-type.

5.6 Cells, source and target. In the following, we call P-cell (of A) an element of A_P for some $A :: \mathcal{U}$ and $P :: \mathsf{PS}$. We also call n-cell a $\mathcal{O}_n$-cell. When c is a P-cell of A, we let its source be $(c_\flat \circ よ(\mathsf{src}_P))^\flat : A_{\partial P}$, and its target $(c_\flat \circ よ(\mathsf{tgt}_P))^\flat : A_{\partial P}$. We denote them respectively $\mathsf{src}(c)$ and $\mathsf{tgt}(c)$. Notice that for any n-cell b of A, there is an *identity* $(n+1)$-cell id_b on b, defined by precomposing b with the n-th suspension of the terminal map $I \to \mathbb{1}$. By construction, it satisfies $\mathsf{src}(\mathsf{id}_b) = \mathsf{tgt}(\mathsf{id}_b) = b$. Remark also that if $c : A_P$ and $f :: A \to B$, then $\mathsf{src}(f_* c) = f_*(\mathsf{src}(c))$ and $\mathsf{tgt}(f_* c) = f_*(\mathsf{tgt}(c))$.

5.7 gluing of two cells. Let $n \in \mathbb{N}$. There is a pasting scheme $[2] = [*, *]$ whose n-th suspension $\$^n[2]$ may be thought as the formal glueing of two $(n+1)$-cells along their n-boundary. For instance, the following illustrate the pasting scheme $\$^2[2]$.



Notice that there is a morphism

$$\mathsf{comp} :\equiv \$^n \left(\begin{matrix} 0 \mapsto 0 \\ 1 \mapsto 2 \end{matrix} , (\mathsf{id}_*, \mathsf{id}_*) \right) : \mathsf{D}_{n+1} \to \$^n[2]$$

such that $\mathsf{src} \circ \mathsf{comp} = \mathsf{src}$ and $\mathsf{tgt} \circ \mathsf{comp} = \mathsf{tgt}$. By assumption, the suspension operation preserves pushout (see 4). And as such, we have the following description of $\langle \$^n[2] \rangle$.

$$\langle \$^n[2] \rangle = \$^n \operatorname{colim} \left(\begin{matrix} & \mathbb{1} & \\ {\scriptstyle \mathsf{right}} \swarrow & & \searrow {\scriptstyle \mathsf{left}} \\ I & & I \end{matrix} \right) = \operatorname{colim} \left(\begin{matrix} & \mathsf{D}_n & \\ {\scriptstyle \mathsf{tgt}} \swarrow & & \searrow {\scriptstyle \mathsf{src}} \\ \mathsf{D}_{n+1} & & \mathsf{D}_{n+1} \end{matrix} \right)$$

Thus, given a crisp type $A :: \mathcal{U}$, any pairs of $(n+1)$-cells $c, d : A_n$ such that $\mathsf{tgt}(c) = \mathsf{src}(d)$ yield a map $\flat(\langle \$^n[2] \rangle \to A)$ which we write (c, d).

> **Definition 5.8 : Codimension-1 composition of cells**
>
> Let $A :: \mathcal{U}$ be a crisp SEGAL-type, and let c, d be two $(n+1)$-cells of A such that $\mathsf{tgt}(c) = \mathsf{src}(d)$. Then by the SEGAL property, the map $(c, d) : \flat(\langle \$^n[2] \rangle \to A)$ extends uniquely to a $\$^n[2]$-cell of A, whose pullback by comp is a $(n+1)$-cell that we denote $c *_n d$. We call $c *_n d$ the *n-composite* of the $(n+1)$-cells c and d.

5.9 In the following, when $a, b : A_n$ are two n-cells of A, we let $a \to_A b :\equiv \sum_{c : A_{n+1}} (\mathsf{src}(c) = a) \times (\mathsf{tgt}(c) = b)$. Note that if this type is inhabited and $n > 0$ then there are identities $\mathsf{src}(a) = \mathsf{src}(b)$ and $\mathsf{tgt}(a) = \mathsf{tgt}(b)$. Definition 5.8 above induces a (crisp) map $(a \to_A b) \times (b \to_A c) \to (a \to_A c)$ when A is a crisp SEGAL-type.

Definition 5.10 : Invertible cell

Let $A :: \mathcal{U}$ be a crisp SEGAL type and $c : a \to_A b$ a $(n+1)$-cell of A. The type of *left-inverses* of c is

$$\mathsf{linv}(c) :\equiv \sum_{d:b\to a} d *_n c = \mathsf{id}_b \quad .$$

The type of *right-inverses* of c is

$$\mathsf{rinv}(c) :\equiv \sum_{d:b\to a} c *_n d = \mathsf{id}_a \quad .$$

Then we let $\mathsf{is\text{-}inv}(c) :\equiv \mathsf{linv}(c) \times \mathsf{rinv}(c)$, and we define the type of *invertible* $(n+1)$-cells of A from a to b as

$$a \simeq_A b :\equiv \sum_{c:a\to_A b} \mathsf{is\text{-}inv}(c) \quad .$$

5.11 Note that for any crisp SEGAL-type A and two n-cells $a, b : A_n$ there is a canonical map $(a =_{A_n} b) \to (a \simeq_A b)$ given by path induction, whose value on refl_a is given by an identity id_a.

Definition 5.12 : (∞, ω)-category

Let $A :: \mathcal{U}$ be a crisp SEGAL type. We say that A is *complete* whenever the canonical maps $a =_{A_n} b \to a \simeq_A b$ are equivalences for all n, a, b.
We also call any complete SEGAL type an (∞, ω)-category and write (∞, ω)-Cat their type.

Lemma 5.13 : First reformulation of completeness

For $A :: \mathcal{U}$ a SEGAL-type, the completeness condition of Definition 5.12 is equivalent to the following one:

$$\forall n, \ \mathsf{is\text{-}equiv}(A_n \to A_{n+1}^{\mathsf{inv}})$$

where $A_{n+1}^{\mathsf{inv}} :\equiv \sum_{a,b:A_n} a \simeq_A b$, and the map $A_n \to A_{n+1}^{\mathsf{inv}}$ is $a \mapsto \mathsf{id}_a$.

Proof. This is directly seen by Theorem 4.7.7 in [30]. □

Lemma 5.14 : Second reformulation of completeness

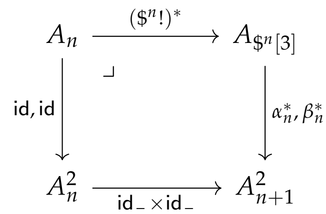
For $n : \mathbb{N}$, let $E_{n+1} : \mathcal{U}$ be the pushout of the following diagram:

$$\begin{array}{ccc} D_{n+1} + D_{n+1} & \xrightarrow{\alpha_n, \beta_n} & \text{よ}\$^n[3] \\ \downarrow & \ulcorner & \downarrow \\ D_n + D_n & \longrightarrow & E_{n+1} \end{array}$$

where the vertical map is on each component the n-th suspension of the terminal map $I \to *$.
And $\alpha_n = \text{よ}(\$^n\alpha)$, $\beta_n = \text{よ}(\$^n\beta)$ where $\alpha, \beta : [1] \to [3]$ are given by

$$\alpha(0) = 0, \quad \alpha(1) = 2 \qquad \text{and} \qquad \beta(0) = 1, \quad \beta(1) = 3$$

Then $A_{n+1}^{\mathsf{inv}} \simeq \flat(E_{n+1} \to A)$ is the pullback $A_n^2 \times_{A_{n+1}^2} A_{\$^n[3]}$. So a SEGAL-type $A :: \mathcal{U}$ is complete iff for each n the following square is cartesian.



Proof. The first statement comes from the fact that an element of $A_{⅄\$^n[3]}$ is equivalently three $(n+1)$-cells $s : a \to_A b$, $f : b \to_A c$ and $p : c \to_A d$, pulling back by α_n, β_n computes the compositions $s *_n f$ and $f *_n p$, and pulling back along $\mathsf{D}_{n+1} + \mathsf{D}_{n+1} \to \mathsf{D}_n + \mathsf{D}_n$ computes two identity $(n+1)$-cells.

The second statement follows from the observation that the map $A_n \to A_{n+1}^{\mathsf{inv}}$ is obtained as the precomposition with the canonical map $E_{n+1} \to D_n$. □

6 Some Results

Discrete types are (∞, ω)-categories

Lemma 6.1

If $A :: \mathcal{U}$ is a discrete crisp type, then for all $a, b :: A$, $\mathsf{hom}_A(a, b) \simeq (a =_A b)$ is a discrete type.

Proof. Since A is discrete, we may see hom_A as a type family $: A \times A \to \mathcal{U}$. We then use Theorem 5.8.4. from [30]. By Postulate 3, we may check the contractibility of $\sum_{b:A} \mathsf{hom}_A(a, b)$ objectwise. For any $P :: \mathsf{PS}$,

$$
\begin{aligned}
(\textstyle\sum_{b:A} \mathsf{hom}_A(a, b))_P &\simeq \textstyle\sum_{f:A_P} \flat\left(\prod_{s:⅄(P)} \mathsf{hom}_A(a, f_\flat\, s)\right) && \text{by Lemma 6.8. in [29].}\\
&\simeq \textstyle\sum_{b:A} \flat \mathsf{hom}_A(a, b) && \text{because } A \text{ is discrete.}\\
&\simeq \textstyle\sum_{b:A} \flat(I \to_{\bullet\bullet} (A, a, b)) && \text{by Postulate 5.}\\
&\simeq \textstyle\sum_{b:A} a = b && \text{because } A \text{ is discrete.}
\end{aligned}
$$
□

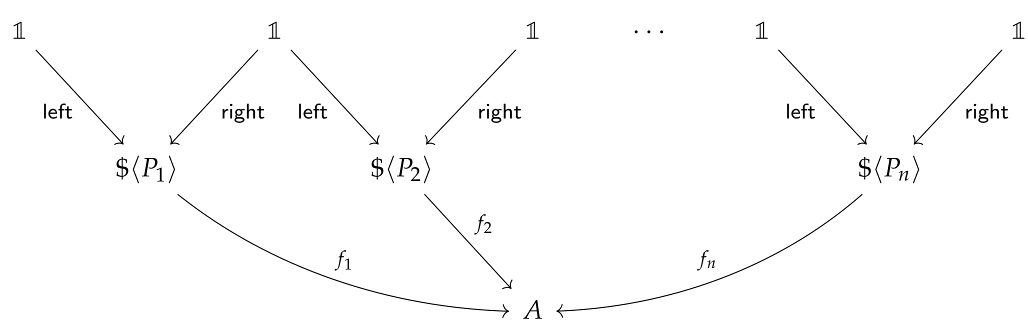
Theorem 6.2

Any crisp discrete type is an (∞, ω)-category.

Proof. Segalness. We show by induction on $P :: \mathsf{PS}$ that for any discrete $A :: \mathcal{U}$, $A \simeq \flat(\langle P \rangle \to A)$ (through constant mapping). Let $P = [P_1, \cdots, P_n]$ and notice that for any i,

$$\flat(\$\langle P_i \rangle \to A) \simeq \flat(\sum_{a,b:A} \langle P_i \rangle \to \mathsf{hom}_A(a, b)) \simeq \sum_{a,b:\flat A} \flat(\langle P_i \rangle \to \mathsf{hom}_A(a, b))$$

and $\mathsf{hom}_A(a, b) \simeq (a = b)$ is discrete by Lemma 6.1. Thus, the right hand side reduces to $\sum_{a,b:A} a = b$ by inductive hypothesis, which is equivalent to A. Hence $\flat(\$\langle P_i \rangle \to A) \simeq A$ through evaluation to any endpoint of $\$\langle P_i \rangle$. Using the universal property of $\langle P \rangle$, a map $\flat(\langle P \rangle \to A)$ is equivalently a cocone



where the data of each f_i is equivalent to the data of an equality between its endpoints $f_i(\mathsf{left}) = f_i(\mathsf{right})$. Hence, the data of such a cocone is equivalent to the data of two points of A and an equality between them, that is, to A. This first part proves that any crisp discrete type $A :: \mathcal{U}$ is a SEGAL-type.

Completeness. Let $A :: \mathcal{U}$ be a crisp discrete type. By discreteness of A, for all $n \in \mathbb{N}$, $A_n \simeq A_*$, so the lower square appearing in Lemma 5.14 is a pullback square. □

Homotopy level is determined objectwise

Lemma 6.3

Postulate 7 may be reformulated as follows:

$$\text{For any } X :: \mathcal{U} \text{ and } P : X \to \mathsf{Prop}_{\mathcal{U}}, \qquad \left(\prod_{R::\mathsf{PS}} \prod_{c:X_R} \prod_{s:よR} P(c_\flat\, s)\right) \to \prod_{x:X} P(x)$$

Theorem 6.4 : Universal property of 0-truncated crisp types

For any $X :: \mathcal{U}$, $\varepsilon : (P, c, s) \mapsto c_\flat(s)$ coequalizes μ, ν as follows:

$$\left(\sum_{P,Q:PS} \sum_{\sigma:P\to_{\mathsf{PS}}Q} \sum_{d:X_Q} よP\right) \underset{\nu}{\overset{\mu}{\rightrightarrows}} \left(\sum_{P:PS} \sum_{c:X_P} よP\right) \xrightarrow{\varepsilon} X$$

where $\mu(P, Q, \sigma, d, s) :\equiv (P, \sigma^* d, s)$ and $\nu(P, Q, \sigma, d, s) :\equiv (Q, d, (よ\sigma)s)$.
Moreover, for any $Y :: \mathsf{Set}_{\mathcal{U}}$, factorizing by ε yield an equivalence

$$(X \to Y) \underset{\varphi}{\xrightarrow{\sim}} \sum_{f:\sum_{P:PS}\sum_{c:A_P} よP \to Y} f \circ \mu = f \circ \nu$$

Proof. For the first assertion, notice that for any P, Q, σ, s, we have $(\sigma^* d)_\flat\, s = (d_\flat \circ よ\sigma)\, s$.

We now prove the second assertion. First, we prove that the fiber of φ is a proposition. Suppose that $f : \sum_{P:PS} \sum_{c:A_P} よP \to Y$ factors as $f^\dagger$ and $f^\ddagger : X \to Y$. Then we show $\prod_{x:X} f^\dagger(x) = f^\ddagger(x)$. Using Lemma 6.3 (we may, because Y is a set), we may show instead:

$$\prod_{R:\mathsf{PS}} \prod_{c:X_R} \prod_{s:よR} f^\dagger(c_\flat\, s) = f^\ddagger(c_\flat\, s)$$

Then, by assumption, both handsides of the equality are identical to $f(R, c, s)$.
Finally, we prove the contractibility of the fiber by constructing a preimage. In order for a factorization $f^\dagger : X \to Y$ of f to be well-defined, we only have to show that $f(P, c, s) = f(Q, d, t)$ whenever $c_\flat s = d_\flat t$. That is $\prod_{v:V} \Phi(v)$ where

$$V :\equiv \sum_{P,Q:\mathsf{PS}} \sum_{c:X_P} \sum_{d:X_Q} \sum_{s:よP} \sum_{t:よQ} c_\flat s = d_\flat t$$

and $\Phi(P, Q, c, d, s, t) :\equiv (f(P, c, s) = f(Q, d, t))$. Φ is a proposition because Y is a set, so we may apply Lemma 6.3 once again. Thus we have to prove $\Phi(m_\flat\, i)$ for some $m : \flat(よR \to V)$ and $i : よR$. Using the discreteness of PS, the idempotence of $\flat$ and the fully-faithfulness of よ we reformulate the goal as:

$$\prod_{R,P,Q:\mathsf{PS}} \prod_{c:X_P} \prod_{d:X_Q} \prod_{s:R\to_{\mathsf{PS}}P} \prod_{t:R\to_{\mathsf{PS}}P} s^* c = t^* d \to f(P, c, よs\, i) = f(Q, d, よt\, i)$$

but then, both handsides of the last equality are identical to $f(R, s^* c, i)$, whence the result. □

Theorem 6.5 : Objectwise equality

For any two maps $f, g :: X \to Y$ where $X :: \mathcal{U}$ and $Y :: \mathsf{Set}_{\mathcal{U}}$,

$$(f = g) \leftrightarrow \prod_{P:PS} \prod_{c:X_P} f_P(c) = g_P(c)$$

Proof. This is a corrollary of Theorem 6.4, asserting that f and g are determined by their images on cells. □

Lemma 6.6

For all n, the n-sphere $\mathbb{S}^n$ is discrete.

Proof. For $\mathbb{S}^{-1} = \mathbb{0}$, this is given by Theorem 6.21. in [29]. Suppose the result to be known for some n, then $\mathbb{S}^{n+1}$ – given by the coequalizer

$$\mathbb{S}^n \underset{1}{\overset{0}{\rightrightarrows}} \mathbb{2} \longrightarrow \mathbb{S}^{n+1}$$

– is again discrete by the theorem *loc.cit.* □

Theorem 6.7

Let $X :: \mathcal{U}$, then for any $n \geq -2$,

$$\mathsf{is}\text{-}n\text{-}\mathsf{type}(X) \leftrightarrow \prod_{P:\mathsf{PS}} \mathsf{is}\text{-}n\text{-}\mathsf{type}(X_P)$$

Proof. The left to right implication is given by Corollary 6.7. in [29].
For the converse implication:

- *Case* $n = -2$.
  The type X is contractible iff $X \to \mathbb{1}$ is an equivalence, which may be tested objectwise by Postulate 3.

- *Case* $n \geq -1$.
  We make use of Theorem 7.2.9. in [30]. We thus have $\mathsf{is}\text{-}n\text{-}\mathsf{type}(X) \leftrightarrow \mathsf{is}\text{-}\mathsf{equiv}\left(X^{\mathbb{S}^{n+1}} \to X\right)$. Now by Postulate 3, this is equivalent to

  $$\prod_{P::PS} \mathsf{is}\text{-}\mathsf{equiv}\left((X^{\mathbb{S}^{n+1}})_P \to X_P\right)$$

  Howether, $(X^{\mathbb{S}^{n+1}})_P \simeq \flat(よP \times \mathbb{S}^{n+1} \to X) \simeq \flat(\mathbb{S}^{n+1} \to X_P)$ where the last equivalence uses Lemma 6.6 from theses notes and Corollary 6.15. from [29]. Finally, $\flat(S^{n+1} \to X_P) \simeq \flat X_P \simeq X_P$ by assumption, whence the equivalence $(X^{\mathbb{S}^{n+1}})_P \simeq X_P$. □

Segalness and completeness are preserved by hom

Lemma 6.8

For any pasting scheme $P : \mathsf{PS}$, there is a commutative square

$$\begin{array}{ccc} \langle \$P \rangle & \xrightarrow{\mathsf{can}_{\$P}} & よ\$P \\ \wr\downarrow & & \downarrow\wr \\ \$\langle P \rangle & \xrightarrow[\$\mathsf{can}_P]{} & \$よP \end{array}$$

Proof. In the general case, the canonical map $\mathsf{can}_Q : \langle Q \rangle \to \text{よ}Q$ is given recursively by the cocone defined in paragraph 5.3. Which, in the case $Q = [P] = \$P$ reduces to

$$\$\langle P \rangle \xrightarrow[\$\mathsf{can}_P]{} \$\text{よ}P \xrightarrow[\sim]{} \text{よ}\$P \xrightarrow[\text{よ}\subseteq_P = \mathsf{id}]{} \text{よ}Q$$

Thus giving the commutative square, by definition of can_Q. □

Lemma 6.9

If $A :: \mathcal{U}$ is a Segal type and $a, b :: A$, then for any pasting scheme P, the canonical map

$$\flat(\text{よ}P \to_{\bullet\bullet} (A, a, b)) \longrightarrow \flat(\langle P \rangle \to_{\bullet\bullet} (A, a, b))$$

is an equivalence

Proof. This is mainly commuting $\flat$ to Σ-types and identity types, and using the bipointedness of can_P. □

Lemma 6.10 : hom preserves Segalness

Let $A :: \mathcal{U}$ be a Segal Type, and $a, b :: A$, then $\mathsf{hom}_A(a, b)$ is also Segal.

Proof.

$$\frac{\flat(\text{よ}P \to \mathsf{hom}_A(a, b))}{\flat(\$'\text{よ}P \to_{\bullet\bullet} (A, a, b))}$$
$$\frac{\flat(\text{よ}\$P \to_{\bullet\bullet} (A, a, b))}{\flat(\langle \$P \rangle \to_{\bullet\bullet} (A, a, b))}$$
$$\frac{\flat(\$'\langle P \rangle \to_{\bullet\bullet} (A, a, b))}{\flat(\langle P \rangle \to \mathsf{hom}_A(a, b))}$$

Where first and last equivalences use the adjunction $\$ \dashv \mathsf{hom}$. the middle one is Lemma 6.9, and the two remaining ones are $\$\text{よ}P \simeq \text{よ}\$P$ and $\langle \$P \rangle \simeq \$\langle P \rangle$. Finally, we make use of Lemma 6.8 to see that this isomorphism of hom types is induced by the canonical map $\langle P \rangle \to \text{よ}P$. □

Theorem 6.11 : hom preserves (∞, ω)-categories

Let $A :: \mathcal{U}$ be a an (∞, ω)-category and $a, b :: A$. Then $\mathsf{hom}_A(a, b)$ is an (∞, ω)-category.

Proof. By Lemma 6.10, we only need to prove that $\mathsf{hom}_A(a, b)$ is a complete type. However, notice that an n-cell $c : \mathsf{hom}_A(a, b)_n \simeq \flat(D_{n+1} \to_{\bullet\bullet} (A, a, b))$ yield a $(n+1)$-cell $\underline{c} : A_{n+1} \simeq \flat(D_{n+1} \to A)$. Hence the completeness of $\mathsf{hom}_A(a, b)$ follows from that of A by shifting indexes and commuting coherences at sources of targets. □

Fibrant realizations are (∞, ω)-categories

Lemma 6.12 : Universal property of suspensions in Θ

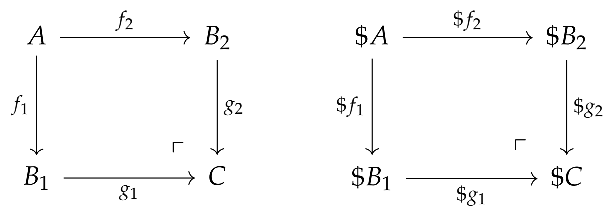
Let $P, Q : \mathsf{PS}$ with $Q = [Q_1, \cdots, Q_n]$, we have

$$\$P \to_{\mathsf{PS}} Q \simeq \sum_{0 \le k \le l \le n} \prod_{k < i \le l} P \to_{\mathsf{PS}} Q_i$$

Proof. This is directly seen by unfolding Definition 3.7. □

Lemma 6.13 : Suspension of pushouts in Θ

If the leftmost square is a pushout square in Θ, then the rightmost too.

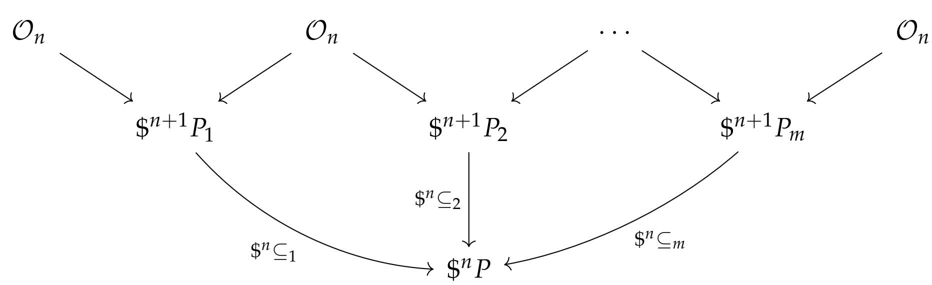


Proof. Let $Q = [Q_1, \cdots, Q_n]$: PS.

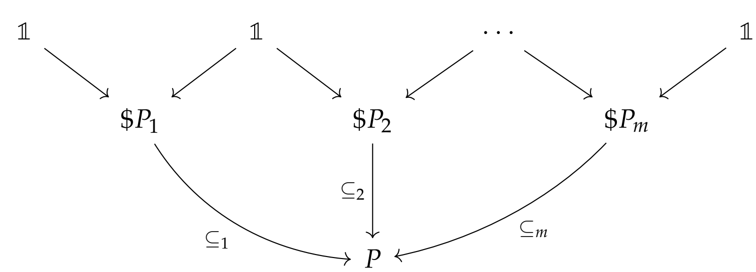
$$
\begin{aligned}
\$C \to_{\mathsf{PS}} Q &\simeq \textstyle\sum_{0\le k\le l\le n} \prod_{k<i\le l} C \to_{\mathsf{PS}} Q_i && \text{by Lemma 6.12.} \\
&\simeq \textstyle\sum_{0\le k\le l\le n} \prod_{k<i\le l} (B_1 \to_{\mathsf{PS}} Q_i) \times_{A\to_{\mathsf{PS}} Q_i} (B_2 \to_{\mathsf{PS}} Q_i) && \text{by assumption} \\
&\simeq \left(\textstyle\sum_{0\le k\le l\le n} \prod_{k<i\le l} B_1 \to_{\mathsf{PS}} Q_i\right) \\
&\qquad \times_{\sum_{0\le k\le l\le n} \prod_{k<i\le l} A\to_{\mathsf{PS}} Q_i} \left(\textstyle\sum_{0\le k\le l\le n} \prod_{k<i\le l} B_2 \to_{\mathsf{PS}} Q_i\right) \\
&\qquad \text{because } \$f_1 \text{ and } \$f_2 \text{ preserve the left and right points.} \\
&\simeq (\$B_1 \to_{\mathsf{PS}} Q) \times_{\$A\to_{\mathsf{PS}} Q} (\$B_2 \to_{\mathsf{PS}} Q) && \text{by Lemma 6.12.}
\end{aligned}
$$
□

Lemma 6.14

Let $P = [P_1, \cdots, P_m]$: PS, the following cocone in Θ is colimiting:



Proof. By unfolding Definition 3.7, we see that the following cocone in colimiting in Θ:



Then we obtain the result by using Lemma 6.13. □

Theorem 6.15 : Representables are SEGAL-types

Let P :: PS, then $よ(P)$ is a SEGAL-type.

Proof. We show by induction on P :: PS the following property:

$$\mathcal{P}(P) \qquad : \qquad \forall n, \forall Q :: \mathsf{PS}, \flat(\$'^n\langle P\rangle \to よ(Q)) \simeq \flat(よ(\$^n P) \to よ(Q))$$

Suppose $P = [P_1, \cdots, P_m]$ such that $\mathcal{P}(P_i)$ holds for every i. Then we have:

$$\begin{aligned}
\flat(\$'^n\langle P\rangle \to_{\mathsf{PS}} よ(Q)) &\simeq \flat(\$'^{n+1}\langle P_1\rangle +_{\mathsf{D}_n} \cdots +_{\mathsf{D}_n} \$'^{n+1}\langle P_m\rangle \to よ(Q)) \\
&\qquad \text{because } \$' \text{ preserves pushouts.} \\
&\simeq \flat(\$'^{n+1}\langle P_1\rangle \to よ(Q)) \times_{\flat(\mathsf{D}_n \to よ(Q))} \cdots \times_{\flat(\mathsf{D}_n \to よ(Q))} \flat(\$'^{n+1}\langle P_m\rangle \to よ(Q)) \\
&\simeq \flat(よ(\$^{n+1}P_1) \to よ(Q)) \times_{\flat(よ(\mathcal{O}_n) \to よ(Q))} \cdots \times_{\flat(よ(\mathcal{O}_n) \to よ(Q))} \flat(よ(\$^{n+1}P_m) \to よ(Q)) \\
&\qquad \text{by inductive hypothesis.} \\
&\simeq \flat(\$^{n+1}P_1 \to Q) \times_{\flat(\mathcal{O}_n \to Q)} \cdots \times_{\flat(\mathcal{O}_n \to Q)} \flat(\$^{n+1}P_m \to Q) \\
&\qquad \text{by fully-faithfulness of } よ \text{ (see Postulate 1).} \\
&\simeq \flat(よ(\$^n P) \to_{\mathsf{PS}} よ(Q)) \\
&\qquad \text{by Lemma 6.14 and fully-faithfulness of } よ.
\end{aligned}$$

Specializing to the case $n = 0$ yield the desired isomorphism. □

Lemma 6.16

Let $P : \mathsf{PS}$ be a pasting scheme, and suppose it has two morphisms $c, d : \mathcal{O}_{n+1} \to_{\mathsf{PS}} P$ such that $c \circ \mathsf{src} = d \circ \mathsf{tgt}$ and $d \circ \mathsf{src} = c \circ \mathsf{tgt}$. Then $c \circ \mathsf{src} = c \circ \mathsf{tgt}$ and $c = d$.

Proof. We write $P = [P_1, \cdots, P_m]$ and proceed by induction on the dimension n.

- *Case* $n = 0$.
  In this case, c induces an inequality $(c \circ \mathsf{src}) \leq (c \circ \mathsf{tgt})$ as elements of $[m]$.
  Similarly $(d \circ \mathsf{src}) \leq (d \circ \mathsf{tgt})$, whence the result.
- *Case* $n > 0$.
  In this case, we may write $c = (f, (\sigma_i)_{i \in I})$ and $d = (g, (\tau_j)_{j \in J})$. First, notice that $f(0) = g(0)$ and $f(1) = g(1)$ because $\mathsf{src} : \mathcal{O}_n \to \mathcal{O}_{n+1}$ preserves endpoints (because $n > 0$). So we deduce $I = J$ and $f = g$. Then, for each $i \in I$, $\sigma_i \circ \mathsf{src} = (c \circ \mathsf{src})_i$ (here we use $n > 0$, see Definition 4.8). And similarly, $\tau_i \circ \mathsf{tgt} = (d \circ \mathsf{tgt})_i$, so $\sigma_i \circ \mathsf{src} = \tau_i \circ \mathsf{tgt}$. Reversing σ_i and τ_i above also yield $\tau_i \circ \mathsf{src} = \sigma_i \circ \mathsf{tgt}$. Hence, by inductive hypothesis, we get $\sigma_i \circ \mathsf{src} = \sigma_i \circ \mathsf{tgt}$ and $\sigma = \tau$. In particular we already have $c = d$. Finally, using the definition of src and tgt again, we have $(c \circ \mathsf{src})_i = (c \circ \mathsf{tgt})_i$ for each i, whence $c \circ \mathsf{src} = c \circ \mathsf{tgt}$.

□

Theorem 6.17

Let $P : \mathsf{PS}$, then $よ(P)$ is an (∞, ω)-category.

Proof. It is a SEGAL-type according to Theorem 6.15. We now see the completeness. On the first hand, notice that if $c, d :: よ(P)_n$, then $c = d$ is propositional by Postulate 1 and Theorem 4.3. On the other hand, using Lemma 6.16, one deduces that if there is an equivalence $c \simeq_{よ(P)} d$, then $c = d$ and this equivalence is unique. So we proved that $c = d$ and $c \simeq d$ are equivalent propositions, which gives the result. □

Remark 6.18
In the proof above, we have achieved a bit more, namely: we have shown that $よ(P)$ is *skeletal*. We have also shown that $\mathbb{1} \simeq よ(*)$ is an (∞, ω)-category (alternatively, it follows from $\mathbb{1}$ being discrete).

Stability under pullbacks and sums

Lemma 6.19 : Cells of pullbacks and sums

Let $A, B, C :: \mathcal{U}$ be three (∞, ω)-categories.

- For any two crisp maps $f :: B \to A$, $g :: C \to A$ and $P :: \mathsf{PS}$,

$$(B \times_A C)_P \simeq B_P \times_{A_P} C_P \quad .$$

- Similarly for sums:
$$(B+C)_P \simeq B_P + C_P \quad .$$

Proof. Pullbacks.

$$\begin{array}{rcll} \flat(よ(P) \to B \times_A C) & \simeq & \flat((よ(P) \to B) \times_{よ(P)\to B} (よ(P) \to C)) & \text{by universal property.} \\ & \simeq & \flat(よ(P) \to B) \times_{\flat(よ(P)\to B)} \flat(よ(P) \to C) & \text{by Theorem 6.10 in [29].} \end{array}$$

Sums. This is given by Postulate 6. □

Lemma 6.20 : Invertible cells of pullbacks and sums

We have the following caracterisation of invertible cells in pullbacks and sums:

- For any $A, B, C :: \mathcal{U}$ SEGAL-types, crisp maps $f :: B \to A$, $g :: C \to A$ and $n : \mathbb{N}$,
$$(B \times_A C)^{\mathsf{inv}}_{n+1} \simeq B^{\mathsf{inv}}_{n+1} \times_{A^{\mathsf{inv}}_{n+1}} C^{\mathsf{inv}}_{n+1} \quad .$$
- For any $B, C :: \mathcal{U}$ SEGAL-types and $n : \mathbb{N}$,
$$(B+C)^{\mathsf{inv}}_{n+1} \simeq B^{\mathsf{inv}}_{n+1} + C^{\mathsf{inv}}_{n+1} \quad .$$

Proof. Pullbacks.

$$\begin{array}{rcll} (B \times_A C)^{\mathsf{inv}}_{n+1} & \simeq & \flat(E_{n+1} \to B \times_A C) & \text{by Lemma 5.14.} \\ & \simeq & \flat((E_{n+1} \to B) \times_{E_{n+1}\to A} (E_{n+1} \to C)) & \text{by universal property.} \\ & \simeq & \flat(E_{n+1} \to B) \times_{\flat(E_{n+1}\to A)} \flat(E_{n+1} \to C) & \text{by Theorem 6.10 in [29].} \\ & \simeq & B^{\mathsf{inv}}_{n+1} \times_{A^{\mathsf{inv}}_{n+1}} C^{\mathsf{inv}}_{n+1} & \end{array}$$

Sums. According to Lemma 6.19, a n-cell f of $A + B$ is either a n-cell of A or a n-cell of B, Then the data of a left (resp. right) inverse of f in $A + B$ will factor through the same component as f. Hence f will be invertible iff it is as a cell of A or as a cell of B. □

Lemma 6.21 : Stability under pullbacks

Let $A, B, C :: \mathcal{U}$ be three (∞, ω)-categories with two crisp maps $f :: B \to A$ and $g :: C \to A$. Then the pullback $B \times_A C$ is again an (∞, ω)-category.

Proof. Segalness. Let $P :: \mathsf{PS}$, we have

$$\begin{array}{rcll} \flat(\langle P\rangle \to B \times_A C) & \simeq & \flat((\langle P\rangle \to B) \times_{\langle P\rangle\to B} (\langle P\rangle \to C)) & \text{by universal property.} \\ & \simeq & \flat(\langle P\rangle \to B) \times_{\flat(\langle P\rangle\to B)} \flat(\langle P\rangle \to C) & \text{by Theorem 6.10 in [29].} \\ & \simeq & B_P \times_{A_P} C_P & \text{by Segalness of } A, B, C. \\ & \simeq & (B \times_A C)_P & \text{by Lemma 6.19.} \end{array}$$

Completeness. We use the reformulation Lemma 5.13 of completeness. Let $n : \mathbb{N}$:

$$\begin{array}{rcll} (B \times_A C)_n & \simeq & (B_n \times_{A_n} C_n)_n & \text{by Lemma 6.19.} \\ & \simeq & (B^{\mathsf{inv}}_{n+1} \times_{A^{\mathsf{inv}}_{n+1}} C^{\mathsf{inv}}_{n+1})_n & \text{by completeness of } A, B, C. \\ & \simeq & (B \times_A C)^{\mathsf{inv}}_{n+1} & \text{by Lemma 6.20.} \end{array}$$ □

Lemma 6.22 : Connectedness of $\langle P\rangle$

Let $P : \mathsf{PS}$ and $X, Y :: \mathcal{U}$, then any map $\flat(\langle P\rangle \to X + Y)$ factors through X or Y.

Proof. Because of Postulate 6, we have that $\text{よ}(P)$ is a *connected object* in the following sense: A type $C :: \mathcal{U}$ is *connected* if for any $X, Y :: \mathcal{U}$, any map $\flat(C \to X + Y)$ factors through X or Y. Now one may check that any pushout of connected types is again connected. By definition (see Definition 5.2) of $\langle P \rangle$, it then follows that $\langle P \rangle$ is also connected. □

Lemma 6.23 : Stability under sums

Let $B, C :: \mathcal{U}$ be two (∞, ω)-categories. Then their sum $B + C$ is again an (∞, ω)-category.

Proof. *Segalness.* Let P : PS, we have

$$\begin{aligned} \flat(\langle P \rangle \to B + C) &\simeq \flat((\langle P \rangle \to B) + (\langle P \rangle \to C)) && \text{by Lemma 6.22.} \\ &\simeq \flat(\langle P \rangle \to B) + \flat(\langle P \rangle \to C) && \text{by Theorem 6.21 in [29].} \\ &\simeq B_P + C_P && \text{by Segalness of } B, C. \\ &\simeq (B + C)_P && \text{by Lemma 6.19.} \end{aligned}$$

Completeness. We use the reformulation Lemma 5.13 of completeness. Let $n : \mathbb{N}$:

$$\begin{aligned} (B + C)_n &\simeq (B_n + C_n)_n && \text{by Lemma 6.19.} \\ &\simeq (B^{\mathsf{inv}}_{n+1} + C^{\mathsf{inv}}_{n+1})_n && \text{by completeness of } B, C. \\ &\simeq (B + C)^{\mathsf{inv}}_{n+1} && \text{by Lemma 6.20.} \end{aligned}$$ □

7 The Subuniverse of Codiscrete Types

7.1 In this section, we explore the properties of the $\sharp$-modality in our setting. We will show that the *codiscrete* types (i.e. $\sharp$-modal) are those whose cells are entirely determined by their 0-skeleton. Thus they have contractible hom-types and are to be thought as a directed counterpart of (-1)-truncated types. Among their properties one has that they form a reflexive subuniverse, and all of them are SEGAL-types.

Definition 7.2 : Codiscrete types

Let $A : \mathcal{U}$. the type A is said to be codiscrete iff it is $\sharp$-modal. That is, when the canonical map $(-)^\sharp : A \to \sharp A$ is an equivalence. We let Codisc denote the type of codiscrete types.

7.3 According to Section 3. (p.20) in the work of M. SHULMAN [29], this is equivalent to $(-)^\sharp : A \to \sharp A$ admitting a retraction. We also mention the following usefull fact about $\sharp$, deduced from the work *loc.cit.*

Theorem 7.4 : Codisc is a subuniverse

Codisc, with the $\sharp$ modality, form a reflexive subuniverse in the sense of Section 7.7 in [30].

Proof. See [29], Section 3. □

7.5 In particular, they are stable under identity-types, dependent sums and product. We now introduce a characterization of codiscrete types in terms of their cells.

Theorem 7.6 : Codiscrete crisp types

Let $A :: \mathcal{U}$ be a crisp type, then A is codiscrete iff for all $P ::$ PS, the canonical map $\flat\text{よ}(P) \to \text{よ}(P)$ induces an equivalence

$$\flat(\flat\text{よ}(P) \to A) \simeq A_P$$

That is, iff the P-cells of A are entirely determined by their 0-skeleton.

Proof. First, suppose that A is $\sharp$-modal, we see that is satisfies the aforementioned property by Corollary 6.26 in [29].

Conversely, suppose that for all P, $\flat(\flat よ(P) \to A) \simeq A_P$. Then we see that $A \to \sharp A$ is an equivalence using Postulate 3, so we show it objectwise. Let $P :: \mathsf{PS}$, then using Corollary 6.26 in *loc.cit.* and the assumption on A:

$$A_P \simeq \flat(\flat よ(P) \to A) \simeq \flat(よ(P) \to \sharp A) \equiv (\sharp A)_P$$

whence $A_P \to (\sharp A)_P$ being an equivalence. ☐

Lemma 7.7

Let $A :: \mathcal{U}$ be a codiscrete crisp type, and $a, b :: A$. Then $\mathsf{hom}_A(a, b) \simeq \mathbb{1}$ is contractible.

Proof. According to Postulate 3, it suffices to show it objectwise. Let $P :: \mathsf{PS}$, we have

$$\begin{aligned}
\mathsf{hom}_A(a,b)_P &\simeq \flat(\$' よ(P) \to_{\bullet\bullet} (A, a, b)) && \text{by Postulate 5.}\\
&\simeq \textstyle\sum_{f:\flat(\$' よ(P) \to A)} \flat(f_\flat(\mathsf{left}) = a) \times \flat(f_\flat(\mathsf{right}) = b) && \text{by Lemma 6.8 in [29].}\\
&\simeq \textstyle\sum_{f:\flat(2 \to A)} \flat(f_\flat(0) = a) \times \flat(f_\flat(1) = b) && \text{by codiscreteness of } A\\
& && \quad\text{(and Corollary 6.26 in \textit{loc.cit.}).}\\
&\simeq \textstyle\sum_{(x,y):(\flat A)^2} (x = a^\flat) \times (y = b^\flat) && \text{by Theorem 6.1. in \textit{loc.cit.}}\\
&\simeq \mathbb{1} && \text{by Lemma 3.11.8 in [30].}
\end{aligned}$$

☐

Lemma 7.8

Let $A :: \mathcal{U}$ be a codiscrete crisp type, then for all $n : \mathbb{N}$,

$$A_{n+1} \simeq A^{\mathsf{inv}}_{n+1}$$

Proof. We use Lemma 5.14 and the definition of E_{n+1} to compute A^{inv}_{n+1}. Because A is codiscrete, using Corollary 6.26 in [29], one has $A^{\mathsf{inv}}_{n+1} \simeq \flat(\flat E_{n+1} \to A)$. So it suffices to show that the map $2 \simeq \flat \mathsf{D}_{n+1} \to \flat E_{n+1}$ is an equivalence. Recall that E_{n+1} is the colimit

$$\mathsf{colim}\left(\mathsf{D}_n \xleftarrow{\$'^n!} \mathsf{D}_{n+1} \xrightarrow{\alpha_n} よ\$^n[3] \xleftarrow{\beta_n} \mathsf{D}_{n+1} \xrightarrow{\$'^n!} \mathsf{D}_n \right)$$

and that, according to Theorem 6.21 in [29], $\flat$ preserves pushouts.

Case $n = 0$. In this case:

$$\flat E_1 \simeq \mathsf{colim}\left(\mathbb{1} \leftarrow 2 \xrightarrow{(0,2)} 4 \xleftarrow{(1,3)} 2 \rightarrow \mathbb{1} \right) \simeq 2$$

hence the map $\flat I \to \flat E_1$ is an equivalence.

Case $n = 1$. Similarly,

$$\flat E_{n+1} \simeq \mathsf{colim}\left(2 \xleftarrow{\mathsf{id}} 2 \xrightarrow{\mathsf{id}} 2 \xleftarrow{\mathsf{id}} 2 \xrightarrow{\mathsf{id}} 2 \right) \simeq 2$$

whence $\flat \mathsf{D}_{n+1} \to \flat E_{n+1}$ being an equivalence. ☐

Lemma 7.9 : Segalness of codiscrete types

Let $A :: \mathcal{U}$ be a codiscrete crisp type. Then A is a SEGAL-type.

Proof. Notice that because $\flat$ preserves pushouts, for any $P = [P_1, \cdots, P_m] :: \mathsf{PS}$,

$$\flat\langle P\rangle \simeq \mathsf{colim}\left(\begin{array}{ccccccccc} \mathbb{1} & & & \mathbb{1} & & & \cdots & & \mathbb{1} \\ & \searrow & \swarrow & & \searrow & \swarrow & & \searrow \; \swarrow & \\ & \flat\text{よ}(\$'P_1) & & & \flat\text{よ}(\$'P_2) & & & \flat\text{よ}(\$'P_m) & \end{array}\right)$$

And for each i, $\flat(\$'P_i) \simeq \mathbb{2}$ by Postulate 4. Hence $\flat(\langle P\rangle) \simeq \mathsf{Fin}_{m+1} \simeq \flat(\text{よ}(P))$. Then, by discreteness of A,

$$\flat(\langle P\rangle \to A) \simeq \flat(\flat\langle P\rangle \to A) \simeq \flat(\flat\text{よ}(P) \to A) \simeq A_P \qquad \square$$

Lemma 7.10

Let $A :: \mathcal{U}$ be a codiscrete crisp type, then it is an (∞, ω)-category iff it is a proposition.

Proof. Because A is discrete, for each $P = [P_1, \cdots, P_m]$, $A_P \simeq \flat(\flat\text{よ}(P) \to A) \simeq A_*^m$. Hence, using Theorem 6.7, A is a proposition iff A_* is.

Now, for $n > 0$, $A^{\mathsf{inv}}_{n+1} \simeq A_{n+1} \simeq A_* \times A_* \simeq A_n$. So the only obstruction to completeness is for $n = 0$. That is, A is complete iff $A^{\mathsf{inv}}_1 \simeq A_0$.

Howether, $A^{\mathsf{inv}}_1 \simeq A_1 \simeq A_* \times A_*$. So A is complete iff the (diagonal) map $A_* \to A_* \times A_* \simeq A^{\mathsf{inv}}_1$ is an equivalence. Which occurs exactly when A_* is propositional, that is iff A is. $\square$

8 (∞, n)-Categories

8.1 In this section we introduce a notion of (∞, n)-category for any $0 \leq n < \omega$, which will be special cases of Definition 5.12. Then we relate this definition to that of (∞, ω)-categories and present some results about them.

Definition 8.2 : (∞, n)-category

Let $A :: \mathcal{U}$ be an (∞, ω)-category. It is said to be an *(∞, n)-category* (for some $n : \mathbb{N}$) iff all its m-cells for $m > n$ are invertible. We also call *∞-groupoid* any *$(\infty, 0)$-category*.

Theorem 8.3 : discrete types are the ∞-groupoids

A crisp type $A :: \mathcal{U}$ is discrete if and only if it is an ∞-groupoid.

Proof. We know from Theorem 6.2 that any discrete type $A :: \mathcal{U}$ is an (∞, ω)-category. By completeness and discreteness we also have $A^{\mathsf{inv}}_{n+1} \simeq A_n \simeq A_{n+1}$ for all n. Hence any discrete type is an ∞-groupoid.

Conversely, suppose $A :: \mathcal{U}$ is an ∞-groupoid. Then by completeness, $A^{\mathsf{inv}}_{n+1} \simeq A_n$, and by definition, $A^{\mathsf{inv}}_{n+1} \simeq A_{n+1}$ for all n. Hence, for all n, $A_{n+1} \simeq A_n$. And more generally this observation holds for any other disrete crisp type. We may now prove that $A_P \simeq A_*$ by induction on P, abstracting over $A :: \mathcal{U}$ any ∞-groupoid. Suppose $P = [P_1, \cdots, P_m]$ such that $A_{P_i} \simeq A_*$ for each i. Then:

$$\begin{array}{rcll} A_P & \simeq & \flat(\langle P\rangle \to A) & \text{by Segalness.} \\ & \simeq & A_{\$'P_1} \times_{A_*} \cdots \times_{A_*} A_{\$'P_m} & \text{because } \flat \text{ preserves pullbacks.} \\ & \simeq & \sum_{a_0, \cdots, a_m : \flat A} \mathsf{hom}_A(a_0, a_1)_{P_1} \times \cdots \times \mathsf{hom}_A(a_{m-1}, a_m)_{P_m} & \text{by } \$' \dashv \mathsf{hom}. \\ & \simeq & \sum_{a_0, \cdots, a_m : \flat A} \mathsf{hom}_A(a_0, a_1)_* \times \cdots \times \mathsf{hom}_A(a_{m-1}, a_m)_* & \text{by inductive hypothesis.} \\ & \simeq & \sum_{a_0, \cdots, a_m : \flat A} \flat(a_0 = a_1) \times \cdots \times \flat(a_{m-1} = a_m) & \text{because } A \text{ is an } \infty\text{-groupoid.} \\ & \simeq & A_* & \end{array}$$

Thus, the map $\flat A \to A$ is an equivalence by Postulate 3. $\square$

Lemma 8.4

Let $A :: \mathcal{U}$ be an (∞, ω)-category. Then it is an $(\infty, n+1)$-category iff all its hom-types are (∞, n)-categories.

Proof. Let $A :: \mathcal{U}$ be a $(\infty, n+1)$-category and $a, b : \flat A$. Then for any $m > n$,

$$
\begin{array}{rcll}
\mathsf{hom}_A(a, b)_m & \simeq & \flat(\mathsf{D}_{m+1} \to_{\bullet\bullet} (A, a, b)) & \text{by } \$' \dashv \mathsf{hom}. \\
& \simeq & \flat(\sum_{f : \mathsf{D}_{n+1} \to A} (f(\mathsf{left}_\flat) = a_\flat) \times (f(\mathsf{right}_\flat) = b_\flat)) & \\
& \simeq & \sum_{f : A_{m+1}} \flat(f_\flat(\mathsf{left}_\flat) = a_\flat) \times \flat(f_\flat(\mathsf{right}_\flat) = b_\flat) & \text{by } \flat \text{ commuting to } \Sigma \text{ and } \times. \\
& \simeq & \sum_{f : A^{\mathsf{inv}}_{m+1}} \flat(f_\flat(\mathsf{left}_\flat) = a_\flat) \times \flat(f_\flat(\mathsf{right}_\flat) = b_\flat) & \text{by hypothesis.} \\
& \simeq & \flat(E_{m+1} \to_{\bullet\bullet} (A, a, b)) & \\
& \simeq & \flat(\$' E_m \to_{\bullet\bullet} (A, a, b)) & \text{because } \$' E_m \simeq E_{m+1} \\
& \simeq & \mathsf{hom}_A(a, b)^{\mathsf{inv}}_m & \text{by } \$' \dashv \mathsf{hom}.
\end{array}
$$

Conversely, suppose $\mathsf{hom}_A(a, b)$ are (∞, n)-categories for all $a, b : \flat A$. Then for any $m > n$

$$
\begin{array}{rcll}
A_{m+1} & \simeq & \flat(\mathsf{D}_{m+1} \to A) & \\
& \simeq & \flat(\sum_{a, b : A} \mathsf{D}_{m+1} \to_{\bullet\bullet} (A, a, b)) & \\
& \simeq & \sum_{a, b : \flat A} \flat(\mathsf{D}_{m+1} \to_{\bullet\bullet} (A, a, b)) & \text{by } \flat \text{ commuting to } \Sigma \text{ and } \times. \\
& \simeq & \sum_{a, b : \flat A} \mathsf{hom}_A(a, b)_m & \text{by } \$' \dashv \mathsf{hom}. \\
& \simeq & \sum_{a, b : \flat A} \mathsf{hom}_A(a, b)^{\mathsf{inv}}_m & \text{by hypothesis.} \\
& \simeq & \sum_{a, b : \flat A} \flat(\$' E_m \to_{\bullet\bullet} (A, a, b)) & \text{by } \$' \dashv \mathsf{hom}. \\
& \simeq & \flat(E_{m+1} \to A) & \text{because } \$' E_m \simeq E_{m+1} \\
& \simeq & A^{\mathsf{inv}}_{m+1} & \qquad\qquad \square
\end{array}
$$

Lemma 8.5

Let $A :: \mathcal{U}$ be a crisp type and $0 \leq n \leq \omega$. Then it is an (∞, n)-category iff:

- It satisfies the n-SEGAL *condition*: For any $P : \mathsf{PS}$, $\mathsf{src}^n : \langle \partial^n P \rangle \to よ(P)$ induce an equivalence
$$A_P \simeq \flat(\langle \partial^n P \rangle \to A)$$
- It is n-*complete*: For any $m < n$, the canonical map $\mathsf{D}_n \to E_{n+1}$ yield an equivalence
$$A_m \simeq A^{\mathsf{inv}}_{m+1}$$

Where src^n is given inductively by the cocone whose legs are the $\$' \langle \partial^n P_i \rangle \to \$' よ(P_i) \to よ(P)$.

Proof. In the case $n = \omega$ there is nothing to prove. We then proceed by induction on $n : \mathbb{N}$.
The case $n = 0$ is given by Theorem 8.3.
We then show the result for $n + 1$ assuming it to hold for n.

Suppose $A :: \mathcal{U}$ is an $(\infty, n+1)$-category.
$(n+1)$-*Segalness*. Let $P = [P_1, \cdots, P_m] :: \mathsf{PS}$. We have

$$
\begin{array}{rcll}
A_P & \simeq & \flat(\langle P \rangle \to A) & \text{by Segalness of } A \\
& \simeq & A_{\$P_1} \times_{A_*} \cdots \times_{A_*} \times A_{\$P_m} & \text{by } \flat \text{ commuting to pullbacks.} \\
& \simeq & \sum_{a_0, \cdots, a_m : \flat A} \mathsf{hom}_A(a_0, a_1)_{P_1} \times \cdots \times \mathsf{hom}_A(a_{m-1}, a_m)_{P_m} & \text{by } \$' \dashv \mathsf{hom}. \\
& \simeq & \sum_{a_0, \cdots, a_m : \flat A} \flat(\langle \partial^n P_1 \rangle \to \mathsf{hom}_A(a_0, a_1)) \times \cdots & \text{by inductive hypothesis} \\
& & \qquad \times \flat(\langle \partial^n P_m \rangle \to \mathsf{hom}_A(a_{m-1}, a_m)) & \quad \text{and Lemma 8.4.} \\
& \simeq & \sum_{a_0, \cdots, a_m : \flat A} \flat(\$' \langle \partial^n P_1 \rangle \to_{\bullet\bullet} (A, a_0, a_1)) \times \cdots & \text{by inductive hypothesis} \\
& & \qquad \times \flat(\$' \langle \partial^n P_m \rangle \to_{\bullet\bullet} (A, a_{m-1}, a_m)) & \quad \text{by } \$' \dashv \mathsf{hom}. \\
& \simeq & \flat(\langle \partial^{n+1} P \rangle \to A) &
\end{array}
$$

$(n+1)$-*completeness*. clear because A is complete as an (∞, ω)-category.

Suppose conversely that A is $(n+1)$-SEGAL and $(n+1)$-complete. First, notice that for any $a, b : \flat A$, $\mathsf{hom}_A(a, b)$ is n-SEGAL. Indeed, for any $P :: \mathsf{PS}$, we have the following chain of equivalences:

$$\begin{array}{rcll} \flat(\langle\partial^n P\rangle \to \mathsf{hom}_A(a, b)) & \simeq & \flat(\$'\langle\partial^n P\rangle \to_{\bullet\bullet} (A, a, b)) & \text{by } \$' \dashv \mathsf{hom}. \\ & \simeq & \flat(\langle\partial^{n+1}\$P\rangle \to_{\bullet\bullet} (A, a, b)) & \\ & \simeq & \flat(よ(\$P) \to_{\bullet\bullet} (A, a, b)) & \text{by } (n+1)\text{-Segalness of } A \\ & \simeq & \flat(よ(P) \to \mathsf{hom}_A(a, b)) & \text{by } \$' \dashv \mathsf{hom}. \end{array}$$

Moreover, $\mathsf{hom}_A(a, b)$ is also n-complete. Indeed, for $m < n$:

$$\begin{array}{rcll} \flat(E_m \to \mathsf{hom}_A(a, b)) & \simeq & \flat(\$'E_m \to_{\bullet\bullet} (A, a, b)) & \text{by } \$' \dashv \mathsf{hom}. \\ & \simeq & \flat(E_{m+1} \to_{\bullet\bullet} (A, a, b)) & \\ & \simeq & \flat(\mathsf{D}_{m+1} \to_{\bullet\bullet} (A, a, b)) & \text{by } (n+1)\text{-completeness of } A \\ & \simeq & \flat(\mathsf{D}_m \to \mathsf{hom}_A(a, b)) & \text{by } \$' \dashv \mathsf{hom}. \end{array}$$

So by inductive hypothesis, all hom-types of A are (∞, n)-categories. So it remain to show (using Lemma 8.4) that A is also an (∞, ω)-category.
Segalness. Let $P = [P_1, \cdots, P_m] :: \mathsf{PS}$,

$$\begin{array}{rcll} \flat(\langle P\rangle \to A) & \simeq & \flat(\$'\langle P_1\rangle \to A) \times_{A_*} \cdots \times_{A_*} \flat(\$'\langle P_m\rangle \to A) & \text{by } \flat \text{ commuting to pullbacks.} \\ & \simeq & \sum_{a_0, \cdots, a_m : \flat A} \flat(\langle P_1\rangle \to \mathsf{hom}_A(a_0, a_1)) \times \cdots & \\ & & \qquad \times \flat(\langle P_m\rangle \to \mathsf{hom}_A(a_{m-1}, a_m)) & \text{by } \$' \dashv \mathsf{hom}. \\ & \simeq & \sum_{a_0, \cdots, a_m : \flat A} \flat(\langle\partial^n P_1\rangle \to \mathsf{hom}_A(a_0, a_1)) \times \cdots & \text{by Segalness and} \\ & & \qquad \times \flat(\langle\partial^n P_m\rangle \to \mathsf{hom}_A(a_{m-1}, a_m)) & \quad n\text{-Segalness of hom-types.} \\ & \simeq & \flat(\$'\langle\partial^n P_1\rangle \to A) \times_{A_*} \cdots \times_{A_*} \flat(\$'\langle\partial^n P_m\rangle \to A) & \text{by } \$' \dashv \mathsf{hom}. \\ & \simeq & \flat(\langle\partial^{n+1} P\rangle \to A) & \\ & \simeq & \flat(よ(P) \to A) \equiv A_P & \text{by } (n+1)\text{-Segalness of } A. \end{array}$$

Completeness. For $k > 1$, the equivalence $A_k \simeq A_{k+1}^{\mathsf{inv}}$ is seen from the hom-types of A being complete. In the case $k = 0$, it follows from A being $(n+1)$-complete. □

Theorem 8.6

If $0 \leq n \leq \omega$, any pullback or sum of (∞, n)-categories is again an (∞, n)-category.

Proof. It follows directly from Definition 8.2, Lemmas 6.21, 6.23 and 6.20. □

9 Directed Homotopy

9.1 In this section we introduce some constructions of directed homotopy theory to motivate further development of this type theory. We postulate a reduced suspension, which should be obtained semantically as a localization of an (∞, ω)-category. Then we show that there is an adjunction $\vec{\Omega} \dashv \vec{\Sigma}$ generalizing that of standard homotopy theory.

Definition 9.2 : Automorphism category

Let $(A, a) :: \mathcal{U}_\bullet$ be a crisp pointed type. We define its *endomorphism* type to be

$$\mathsf{End}_A(a) :\equiv \mathsf{hom}_A(a, a)$$

By analogy with the loopspace construction of [30], we also denote it $\vec{\Omega}(A, a)$

9.3 Note that by Lemma 6.10 (resp. Lemma 8.5), if A is a SEGAL-type (resp. an $(\infty, n+1)$-category) then $\vec{\Omega}(A, a)$ is again a SEGAL-type (resp. an (∞, n)-category). In the next definition, we consider a new kind of higher inductive type, which should be thought as a localization of a category.

Definition 9.4 : Reduced suspension

We postulate, for $(A, a) :: \mathcal{U}_\bullet$ a pointed (∞, ω)-category, its reduced suspension $\overrightarrow{\Sigma}(A, a) :: \mathcal{U}_\bullet$. It is caracterised by the following universal property.

- It is an (∞, ω)-category.
- There is a crisp pointed map $\mathsf{loc}_{A,a} :: (\$'A, \mathsf{left}) \to_\bullet \overrightarrow{\Sigma}(A, a)$ such that $\mathsf{is\text{-}inv}(f_*\mathsf{merid}(a))$.
- For any pointed (∞, ω)-category $(B, b) :: \mathcal{U}_\bullet$ and crisp pointed map $f :: (\$'A, \mathsf{left}) \to_\bullet (B, b)$ such that $\mathsf{is\text{-}inv}(f_*\mathsf{merid}(a))$, f factors uniquely through $\mathsf{loc}_{A,a}$.

9.5 The idea is that the reduced suspension $\overrightarrow{\Sigma}(A, a)$ is a *localization* of $\$'A$ at the 1-cell merid_a. Using this functor, one may construct an analogous of n-spheres, namely *directed n-spheres* as follows.

- $\overrightarrow{\mathbb{S}}^0 :\equiv \mathbb{2}$.
- $\overrightarrow{\mathbb{S}}^{n+1} :\equiv \overrightarrow{\Sigma}(\overrightarrow{\mathbb{S}}^n)$.

Definition 9.6 : Unit $\eta_{(A,a)} : \flat((A, a) \to \overrightarrow{\Omega}\,\overrightarrow{\Sigma}(A, a))$

For all $(A, a) :: \mathcal{U}_\bullet$ a pointed (∞, ω)-category, there is a canonical map $\eta_{(A,a)} : (A, a) \to_\bullet \overrightarrow{\Omega}\,\overrightarrow{\Sigma}(A, a)$. Which is given by composing the unit $A \to \mathsf{hom}_{\$'A}(\mathsf{left}, \mathsf{right})$ of the $\$' \dashv \mathsf{hom}$ adjunction with the map $\mathsf{hom}_{\$'A}(\mathsf{left}, \mathsf{right}) \to \mathsf{hom}_{\overrightarrow{\Sigma}(A,a)}(\mathsf{loc}\ \mathsf{left}, \mathsf{loc}\ \mathsf{left})$ induced by the localization $\mathsf{loc}_{A,a}$.

9.7 Notice that the last map is not entirely trivial. We shall mention that in general – in our type theory – there is no map $\mathsf{hom}_X(x, y) \to \mathsf{hom}_X(x, z)$ arising from a 1-cell $y \to z$ in X. Indeed, we may only define this map objectwise

$$\mathsf{hom}_X(x, y)_P \to \mathsf{hom}_X(x, z)_P \quad (P :: \mathsf{PS})$$

and there is no way to wrap it into a map $\mathsf{hom}_X(x, y) \to \mathsf{hom}_X(x, z)$ as this would require giving higher coherences (naturality squares for each morphism $P \to_{\mathsf{PS}} Q$ in Θ, and coherences between these for any two composables morphisms of Θ and so on...) And, to be honnest, this is a major issue with this type theory in its current state. Howether, in the specific case where we have an equality $y =_{\flat X} z$, then we may transport along this equality, which gives us the sought map $\mathsf{hom}_X(x, y) \to \mathsf{hom}_X(x, z)$. Since the localization $\overrightarrow{\Sigma}(A, a)$ is assumed to be an (∞, ω)-category, it is complete, so we know that there is an equality

$$\mathsf{loc}\ \mathsf{left} =_{\flat \overrightarrow{\Sigma}(A,a)} \mathsf{loc}\ \mathsf{right}$$

witnessing the invertibility of $\mathsf{loc}_*\ \mathsf{merid}(a)$. Thus we may transport along this identity, whence a map

$$\mathsf{hom}_{\overrightarrow{\Sigma}(A,a)}(\mathsf{loc}\ \mathsf{left}, \mathsf{loc}\ \mathsf{right}) \to \mathsf{hom}_{\overrightarrow{\Sigma}(A,a)}(\mathsf{loc}\ \mathsf{left}, \mathsf{loc}\ \mathsf{left}) \quad .$$

From now on, we will write $\mathsf{loc}(\mathsf{left})$ (resp. $\mathsf{loc}(\mathsf{right})$ and $\mathsf{loc}(\mathsf{merid}(a))$) more concisely as left, right and $\mathsf{merid}(a)$.

Theorem 9.8 : $\overrightarrow{\Sigma} \dashv \overrightarrow{\Omega}$

Writting $(\infty, \omega)\text{-}\mathsf{Cat}_\bullet$ for the type of pointed (∞, ω)-categories, there is an adjunction

$$(\infty, \omega)\text{-}\mathsf{Cat}_\bullet \;\underset{\overrightarrow{\Omega}}{\overset{\overrightarrow{\Sigma}}{\rightleftarrows}}\; (\infty, \omega)\text{-}\mathsf{Cat}_\bullet \qquad (\bot)$$

In the following sense: For any two crisp pointed types $(A, a), (B, b) :: \mathcal{U}_\bullet$, there is an equivalence

$$\begin{aligned} \flat\Big(\overrightarrow{\Sigma}(A, a) \to_\bullet (B, b)\Big) &\longrightarrow \flat\Big((A, a) \to_\bullet \overrightarrow{\Omega}(B, b)\Big) \\ f &\longmapsto (\overrightarrow{\Omega}(f_\flat) \circ \eta_{A,a})^\flat \end{aligned}$$

Proof.

$$\frac{\dfrac{\dfrac{\flat(\overrightarrow{\Sigma}(A, a) \to_\bullet (B, b))}{f : \flat((\$'A, \mathsf{left}) \to_\bullet (B, b)) \text{ s.t. } \mathsf{is\text{-}inv}(f_*\mathsf{merid}(a))}}{f : \flat((\$'A, \mathsf{left}, \mathsf{right}) \to_{\bullet\bullet} (B, b, b)) \text{ with } f_*\mathsf{merid}(a) = \mathsf{id}_a}}{\flat((A, a) \to_\bullet \overrightarrow{\Omega}(B, b))}$$

□

Lemma 9.9 : Representability of $\overrightarrow{\Omega}$

Let $(A, a) :: (\infty, \omega)\text{-}\mathsf{Cat}_\bullet$, there is an equivalence

$$\flat(\overrightarrow{\mathbb{S}}^1 \to_\bullet (A, a)) \simeq \flat\overrightarrow{\Omega}(A, a)$$

Proof. This follows directly from paragraph 9.5 and Theorem 9.8. □

10 Perspectives and Conclusion

10.1 On a presheaf type theory. One of the main features of this type theory is the ability to speak about a higher category of presheaves over the category Θ. In fact, it seems that most of this work does not rely crucially on the category Θ, namely, postulates 1, 2, 3, 6 and 7, would find a semantic in any suitable category $\mathcal{C}$. At least for a locally finite category which could be internalised in MLTT. This remark raises the prospect of similar type theory, finding semantics in many presheaf ∞-topos. For instance, one could mimic this construction to speak about other models of ∞-categories, or other higher algebraic structures such as Γ-spaces.

10.2 On functoriality. Although our approach has allowed us to define basics constructions of higher category theory, it crucially lacks the "free" functoriality that should be the directed counterpart of the homotopy invariance of every construction made in standard HoTT. For instance, the types of equalities $x =_A y$ between two elements of A comes equiped with a functoriality in x, y given by the transport along other paths of A. More concretely, given $p : y =_A z$, one gets a *transport* map $x =_A y \to x =_A z$. This is a non-directed account for the concatenation of 1-cells. Of course we should expect a similar pattern to occur in our directed variant, that is, for any $x, y, z : A$ and $p : \mathsf{hom}_A(y, z)$, a map $\mathsf{hom}_A(x, y) \to \mathsf{hom}_A(y, z)$. But the type $\mathsf{hom}_A(x, y)$ is not even defined for $x, y : A$. But we expect that hom_A could be seen as a map

$$\mathsf{hom}_A : A \times A^{\mathrm{op}} \to \mathcal{U}$$

which would, then, be functorial in its arguments. To solve this issue – and seemingly several related ones – it seems to the authors that working with objects of *lax functors* $X \rightharpoonup Y$ between types X, Y would gives a lot more freedom. Recall, as pointed in paragraph 1.3, that if A is an (∞, ω)-category, the type $I \to A$ should not be seen as the correct space of 1-cells in A because it does not capture the higher cells transiting between the 1-cells of A. Howether, it seems that $I \rightharpoonup X$, as a right adjoint to the CRANS-GRAY tensor product $Y \otimes I$ would be a better behaved alternative. And that the correct notions of slices or hom-types could be carved out of this type of lax functors $I \rightharpoonup X$.

10.3 The CRANS-GRAY tensor product. As mentioned in the previous paragraph, there is an alternative to the cartesian product of categories called the CRANS-GRAY tensor product $\otimes$. which insert higher directed cells instead of higher isomorphisms in the squares appearing in $X \otimes Y$. The archetypical example is the following one. On the left, we represent the usual cartesian product $I \times I$, with its degenerated triangle. While on the right, we depict their CRANS-GRAY product $I \times I$.

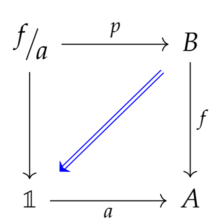
The right one indeed is a 2-categorical object, while the usual cartesian product $I \times I$ remains 1-categorical (as made precise in Theorem 8.6). This nature correct the defect mentioned in paragraph 1.3 from the introduction, and explains why a right adjoint $(I \rightharpoonup -)$ to $(- \otimes I)$ should capture the correct higher cells. More over, it is expected that the suspension $\$'$ we have postulated should be recovered from the tensor product, giving a further motivation for including it in our type theory. It is finally also expected that the notions of *(co)cartesian fibration* would be definable using $\otimes$, thus opening the way towards a formalization of the (∞, ω)-categorical YONEDA lemma.

10.4 A bunched variation of CellTT. As mentioned in paragraph 10.3, we would like to include in our type system new binary operations. At least the CRANS-GRAY tensor product $(- \otimes -)$, together with the type of lax morphisms $(- \rightharpoonup -)$ as its right adjoint, and perhaps even a type of *oplax functors* $(- \rightharpoondown -)$ as a left adjoint. This raises several difficulties, as now the two differents products $\times$ and $\otimes$ should be reflected in context comprehension operations. Concretely, in our cases, we would expect the grammar of context to be at least as rich as:

$$\Gamma, \Delta \quad ::= \quad \diamond \mid \Gamma * \Delta \mid \Gamma, x : A$$

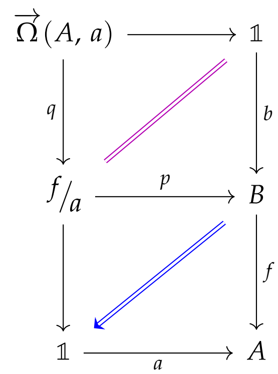
This additional complexity naturally leads us to the lands of bunched logic, historically introduced by D. J. PYM and P. W. O'HEARN [21], then studied by U. SHOËPP [27] and more recently revisited by M. RILEY [26]. The most difficult part seems to lays in the entanglement of bunched logic and dependent types, although in our case, it is expected that dependancy in $\otimes$-types would not be needed (or what even should be its semantics ?). We should also mention that it would leads to a rather specific kind of bunched logic, because $\otimes$ is both non-symmetric, and semi-cartesian. The litterature on bunched logic seems to cover mostly symmetric monoidal products at the moment so this would require some new theory, although major complications are not foreseen on the side of non-commutativity. On the other hand, semi-cartesianness should facilitate our task, as it allows one to discard ressources. More specifically, we anticipate some simplification regarding the complexity of type checking, which is one of the major drawbacks of a bunched typing system.

10.5 Towards directed homotopy theory. Finally, we would like to include a small picture of what should be available with a theory including the CRANS-GRAY tensor product and types of lax maps. First, as mentionned earlier, we might define slice types ${}^{A}\!/_{a}$ for an (∞, ω)-category A and any point a of A, and more generally comma types ${}^{f}\!/_{a}$. These types should fit in a *comma square*, which is a variant of a pullback square, where the commutativity has been replaced by a natural transformation as depicted below.

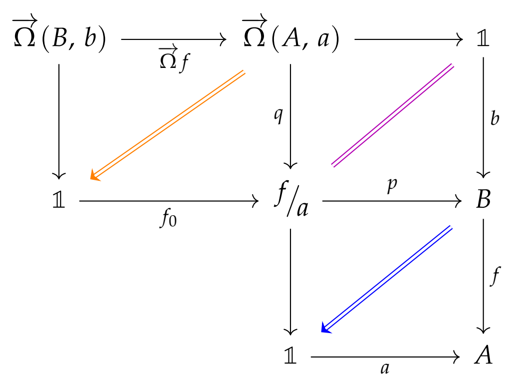


Now, suppose $f : (A, a) \to_{\bullet} (B, b)$ is a *pointed map* in the sense that it is equiped with an identity $f_0 : f\ b = a$.

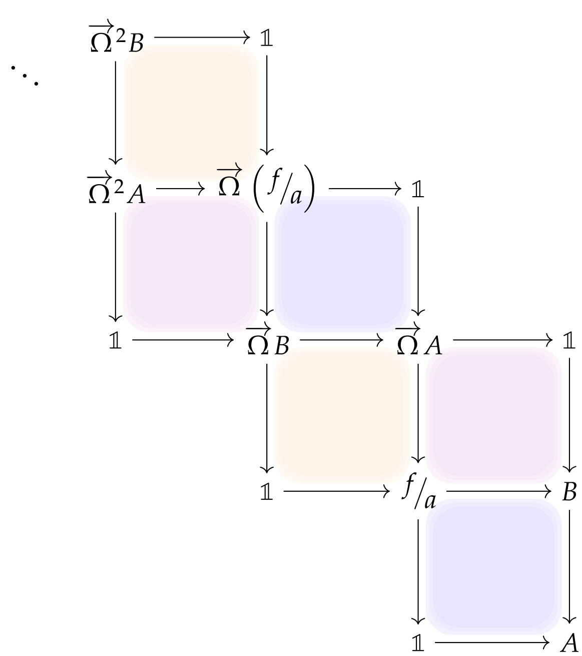
Then one may consider the (usual, homotopy) fiber of $p : {}^{f}/_{a} \to B$, as depicted below.



Then, by a variation on the *pullback pasting lemma*, it is expected that the rectangle is again a comma, thus yielding the *directed* loopspace $\vec{\Omega}(A, a) = \mathsf{hom}_A(a, a)$. Now, one may formally insert to this picture a third square (*a priori* not a comma) by considering the directed loopspace $\vec{\Omega}(B, b)$, which makes the large horizontal rectangle into another comma.



Now, by making the further observation that $\vec{\Omega}\left({}^{f}/_{a}\right) \simeq {}^{\vec{\Omega} f}/_{\mathsf{id}_a}$, one may carry on this procedure, leading to the following (informal picture), generalizing the sequence appearing in (non-directed) homotopy theory.



In general, one cannot expect the orange squares appearing in this picture to have a specific property. But if f is a fibration in a suitable sense (that is if it allows to lift cells from A to B), then the orange squares are indeed expected to be behaves as pullback or commas depending on the nature of the fibration f. Now, taking the 0-truncation of this diagram should yield a long sequence

$$\cdots \to \vec{\pi_2}(B) \to \vec{\pi_2}(A) \to \vec{\pi_1}\left({}^{f}/_{a}\right) \to \vec{\pi_1}(B) \to \vec{\pi_1}(A) \to \vec{\pi_0}\left({}^{f}/_{a}\right) \to \vec{\pi_0}(B) \to \vec{\pi_0}(A)$$

Where the object appearing are posets for the three extremal ones, then *monoidal posets* for the further ones. That is, monoids equiped with a preorder, preserved by the inner product. Where the degree of exactness depends on the position in the sequence. For instance, it should be exact at purple points, and *lax exact* at blue points. Which means that the image of a map will be the set of negative points in the following one.

The authors hope that such kinds of constructions would open the way to computation of directed homotopy monoids of higher categories such as the directed spheres mentioned earlier in this paper.

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A Semantics of CellTT

A.1 In this Appendix, we sketch the semantics of the type theory we have introduced, thus providing motivation for its axioms. We consider a model in *simplicial presheaves over* Θ, equipped with the REEDY model structure, relative to the QUILLEN model structure on $\hat{\Delta}$. Which is, according to J.E. BERGNER and C. REZK [6], the same as the injective model structure. We denote $\mathsf{sPsh}(\Theta)$ this model category. We will denote $\simeq$ (a zig-zag of) weak equivalences and $\cong$ an isomorphism in a model category. We let $よ : \Theta \hookrightarrow \hat{\Theta} \hookrightarrow \mathsf{sPsh}(\Theta)$ denote the YONEDA embedding and recall the following properties of this model structure, which may be found in [28, 24, 11, 18].

- Any objectwise discrete object of $\mathsf{sPsh}(\Theta)$ (*i.e.* set valued) is a fibrant object.
- Any object of $\mathsf{sPsh}(\Theta)$ is cofibrant, cofibrations are the monomorphisms.
- Any two objects X, Y of $\mathsf{sPsh}(\Theta)$ have a *mapping space* $\mathsf{Map}(X, Y) \in \hat{\Delta}$ and an *internal* Hom denoted $\underline{\mathsf{Hom}}(X, Y) \in \mathsf{sPsh}(\Theta)$. Moreover, $\mathsf{Map}(よ(P) \times X, Y) = \underline{\mathsf{Hom}}(X, Y)_P$ for any $P \in \Theta$, and $\mathsf{Map}(X, Y) = \underline{\mathsf{Hom}}(X, Y)_*$ are the global sections of $\underline{\mathsf{Hom}}(X, Y)$. $\mathsf{Map}(X, Y)$ is fibrant as soon as Y is because every object is cofibrant in $\mathsf{sPsh}(\Theta)$.
- A map $f : A \to B$ in $\mathsf{sPsh}(\Theta)$ is a fibration iff for each $P \in \mathcal{O}b(\Theta)$ the following maps is a KAN fibration
$$A_P \to B_P \times_{M_P B} M_P A$$
Where $M_P X = \lim_{(\Theta\downarrow P)^{\mathrm{op}}}(X|_{(\Theta\downarrow P)^{\mathrm{op}}})$ is the *matching object* of X. This holds more generally for another REEDY category $\mathcal{R}$ instead of Θ.
- It is a model of MLTT with dependent sums, dependent product, identity types, pushout types, truncations and a univalent universe for each innaccessible cardinal above $\aleph_0$.

We will call *set* any discrete simplicial set. Note that limits and colimits being computed objectwise implies that sets are stable by limits and colimits.

> **Lemma A.2**
>
> Let $f : A \to B$ be a map between set valued presheaves in $\mathsf{sPsh}(\Theta)$. Then it is a fibration.

Proof. First, notice that if X is set valued, then for any P, $M_P X$ is a set as a limit of sets. Now, if A and B are set valued, this argument shows that $B_P \times_{M_P B} M_P A$ is also a set (again taking a limit of sets). Hence, the map $A_P \to B_P \times_{M_P B} M_P A$ is a KAN fibration as a map between discrete simplicial sets. □

Postulate 1

A.3 Semantic of PS. According to Lemma A.2, any objectwise discrete presheaf is a fibrant object. Hence, we may model the type PS of pasting schemes as the constant presheaf $[\![\mathsf{PS}]\!] := P \mapsto \mathcal{O}b(\Theta)$, which is objectwise discrete. Because it is constant, it agrees with our remark made in paragraph 4.12 about PS being $\flat$-discrete.

A.4 Semantic of $P \to_{\mathsf{PS}} Q$. We should model the types $P \to_{\mathsf{PS}} Q$ as a fibration over $[\![\mathsf{PS}]\!]$. By Lemma A.2, it suffices to model $P \to_{\mathsf{PS}} Q$ for each $P, Q \in [\![\mathsf{PS}]\!]$, which may be given by $[\![P \to_{\mathsf{PS}} Q]\!] := \mathsf{Hom}_{\Theta}(P, Q)$.

A.5 Semantic of $\text{よ} : \mathsf{PS} \to \mathcal{U}$. We now consider the YONEDA embedding postulated in Postulate 1. Once again using Lemma A.2, it suffices to give an interpretation $[\![\text{よ}P]\!]$ as a set valued presheaf for each P. Which we define as $[\![\text{よ}P]\!] := \text{よ}(P)$, the representable functor associated to P. $[\![\text{よ}]\!]$ is now given by the fibration $\coprod_{P \in \mathcal{O}b(\Theta)} \text{よ}(P) \twoheadrightarrow \mathcal{O}b(\Theta)$.

We also have to give, for each $\sigma : P \to_{\mathsf{PS}} Q$, a map $[\![\text{よ}(\sigma)]\!] : \text{よ}(P) \to \text{よ}(Q)$, which is gievn by $\text{よ}(\sigma)$. So the semantic of よ is really given by the YONEDA embedding. In particular, because $\text{よ} : \Theta \to \mathsf{sPsh}(\Theta)$ is functorial, it shows that the equality rules postulated in Postulate 1 holds on the nose. So they could even be postulated as strict equalities.

For P and Q, $[\![P \to_{PS} Q]\!]$ is a constant (and set valued) presheaf with value $\mathsf{Hom}_{\Theta}(P, Q)$, and $\flat(\text{よ}(P) \to \text{よ}(Q))$ is interpreted as the constant presheaf whose value is $\underline{\mathsf{Hom}}(\text{よ}(P), \text{よ}(Q))_*$. Then both coincides since

$$\underline{\mathsf{Hom}}(\text{よ}(P), \text{よ}(Q))_* = \mathsf{Map}(\text{よ}(P), \text{よ}(Q)) \simeq \mathsf{Hom}_{\Theta}(P, Q)$$

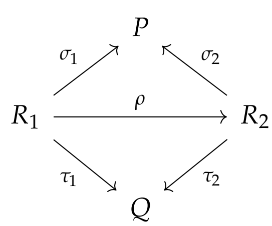
by the YONEDA lemma, so this justifies the last point of Postulate 1.

Postulate 2

A.6 Discreteness. Recall that $\flat$ is interpreted as the map $X \mapsto (P \mapsto X_*)$ which turns a presheaf X into the constant one whose value is X_*. Hence a type interpreted as X in $\mathsf{sPsh}(\Theta)$ is $\flat$-discrete whenever the map $\flat X \to X$ is a weak equivalence. That is, when for any $P \in \mathcal{O}b(\Theta)$, the map $x \mapsto !^*x : X_* \to X_P$ is weak equivalence of simplicial sets. On the other hand, the same type will be cellularly discrete whenever the maps $X_Q \to \mathsf{Map}(\text{よ}(P) \times \text{よ}(Q), X)$ are weak equivalences for each P and Q. By specializing this last condition to $Q = *$, we see that cellular discrete types are $\flat$-discrete. We will then focus on the converse implication.

Definition A.7

Let $P, Q \in \mathcal{O}b(\Theta)$. We denote $\mathcal{D}_{P,Q}$ (or $\mathcal{D}$ for short) the category of elements of $\text{よ}(P) \times \text{よ}(Q) \in \hat{\Theta}$. Hence $\mathcal{D}$ is a category whose objects are the triplets $P \xleftarrow{\sigma} R \xrightarrow{\tau} Q$, and morphisms are the commutative diagrams:



And $F_{P,Q} : \mathcal{D} \to \mathsf{Psh}(\Theta) \hookrightarrow \mathsf{sPsh}(\Theta)$ is the diagram sending such a morphism in $\mathcal{D}$ to $\text{よ}(\rho) : \text{よ}(R_1) \to \text{よ}(R_2)$.

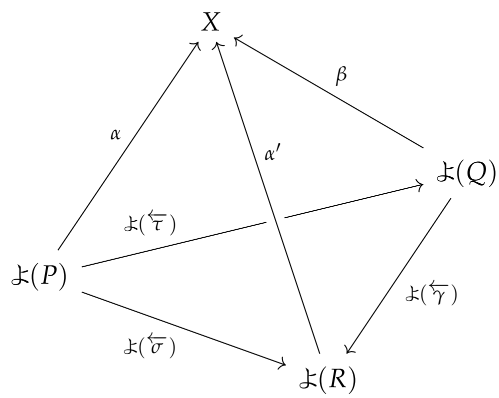
Lemma A.8

The category $\mathcal{D}$ is REEDY and has fibrant constants (see 15.10 in [11]).

Proof. The category $\mathcal{D}$ is REEDY as a category of elements of a presheaf over the REEDY category Θ. More precisely, the degree of $P \leftarrow R \to Q$ is defined as the degree of R, and the increasing (resp. decreasing) morphsms in $\mathcal{D}$ are those which are increasing (resp. decreasing) in Θ.

In order to see that $\mathcal{D}$ has fibrant constant, we use the Proposition 15.10.2.(2) in [11]. That is, we must show that each category $\partial(\alpha \downarrow \overleftarrow{\mathcal{D}})$ is empty or connected. Let X be the presheaf $\text{よ}(P) \times \text{よ}(Q)$ (seen as a set valued presheaf), we will in fact show this result for X any such presheaf and $\mathcal{D}$ its category of elements. Suppose $\alpha : X_P$ is a P-cell of X. Then an object of $\partial(\alpha \downarrow \overleftarrow{\mathcal{D}})$ is the choice of a decomposition $\alpha = \overleftarrow{\sigma}^* \alpha'$ for some σ a strictly decreasing morphism of Θ and α' another cell of X.

- If α is non-degenerate, then there is no-such decomposition, hence $\partial(\alpha \downarrow \overleftarrow{\mathcal{D}})$ is empty.
- In the other case, there is a unique decomposition $\alpha = \overleftarrow{\sigma}^* \alpha'$ where α' is non-degenerate and σ strictly decreasing. Hence, for any other object $\alpha = \overleftarrow{\tau}^* \beta$ in $\partial(\alpha \downarrow \overleftarrow{\mathcal{D}})$, we may decompose uniquely β as $\overleftarrow{\gamma}^* \beta'$ where β' is non-degenerate and $\overleftarrow{\gamma}$ is decreasing. By uniqueness of such decompositions, it follows that $\overleftarrow{\sigma} = \overleftarrow{\gamma} \circ \overleftarrow{\tau}$ and $\beta' = \alpha'$. That is, there is a (unique) commutative diagram



Hence, the decomposition $\alpha = \overleftarrow{\sigma}^* \alpha'$ is a terminal object in $\partial(\alpha \downarrow \overleftarrow{\mathcal{D}})$, so the matching category is connected. □

Lemma A.9

Let $R \in \Theta$ and consider the colimit

$$\partial R := \operatorname{colim}_{(Q \to R) \in \partial(\overrightarrow{\Theta} \downarrow R)} \text{よ}(Q)$$

Then the canonical map $\partial R \to \text{よ}(R)$ is a monomorphsim, identifying ∂R with the subpresheaf of morphisms $f : Q \to R$ which factors through a strictly increasing morphism.

Proof. Because colimits are computed objectwise and monomorphisms are alo objectwise, we show that the map $(\partial R)_P \to \text{よ}(R)_P$ is injective for each $P \in \mathcal{O}b(\Theta)$. We use the explicit description of $(\partial R)_P$ as

$$\frac{\coprod_{\overrightarrow{\sigma} : Q \to R} \mathsf{Hom}_\Theta(P, Q)}{(\overrightarrow{\sigma} \circ \overrightarrow{\tau}, f) \sim (\overrightarrow{\sigma}, \overrightarrow{\tau} \circ f) \quad (\tau \in \overrightarrow{\Theta})}$$

Such that the canonical map $(\partial R)_P \to \text{よ}(R)_P$ sends the class $[(\overrightarrow{\sigma}, f)]$ to $\overrightarrow{\sigma} \circ f$. Then we see that the image of $(\partial R)_P$ is the set of morphisms $P \to R$ which factors through a strictly increasing morphism $\overrightarrow{\sigma}$.

We now show the injectivity. Suppose there is a morphism $h : P \to R$ such that $h = \overrightarrow{\sigma} \circ f = \overrightarrow{\tau} \circ g$ for some strictly increasing morphisms $\overrightarrow{\sigma}$ and $\overrightarrow{\tau}$. Then factorizing f as $\overrightarrow{f} \circ \overleftarrow{f}$ and g as $\overrightarrow{g} \circ \overleftarrow{g}$, we get $\overrightarrow{\sigma} \circ \overrightarrow{f} = \overrightarrow{\tau} \circ \overrightarrow{g}$ and $\overleftarrow{f} = \overleftarrow{g}$ by uniqueness of the factorization of h. Hence, there is a chain of identifications

$$(\overrightarrow{\sigma}, f) = (\overrightarrow{\sigma}, \overrightarrow{f} \circ \overleftarrow{f}) \sim (\overrightarrow{\sigma} \circ \overrightarrow{f}, \overleftarrow{f}) = (\overrightarrow{\tau} \circ \overrightarrow{g}, \overleftarrow{g}) \sim (\overrightarrow{\tau}, \overrightarrow{g} \circ \overleftarrow{g}) = (\overrightarrow{\tau}, g) \quad \square$$

Lemma A.10

The nerve $\mathsf{N}\mathcal{D}$ of the category $\mathcal{D}$ is contractible.

Proof. Recall that the category $\mathcal{D}_{P,Q}$ is the comma (or the category of elements) ${}^{\Theta}/_{\text{よ}(P) \times \text{よ}(Q)}$. Then, thanks to the hard work of D-C. CISINSKI and G. MALTSINIOTIS in [8], it is know that the category Θ is a strict test category (Exemple 5.12 in *loc.cit.*). Thus, by Theorem 2.8 in *loc.cit.* it is totally aspherical, which implies precisely the contractibility of $\mathcal{D}_{P,Q}$. □

Theorem A.11

Both notions of discreteness coincide.

Proof. We already have seen the first direction in paragraph A.6, so we focus on the other one. Suppose that $\flat X \to X$ is a weak equivalence for some fibrant object X (we assume X to be fibrant because it models a type in the empty context). Then, for each $P \in \mathcal{O}b(\Theta)$, we have a weak equivalence $X_* \to X_P$. Recall that we want to show that the map $X_* \to \mathsf{Map}(\text{よ}(P) \times \text{よ}(Q), X)$ is a weak equivalence.

We have seen (Lemma A.8) that $\mathcal{D}$ is REEDY with fibrant constants. Moreover, for any object $\alpha = (P \leftarrow R \to Q)$, the latching category $\partial(\overrightarrow{\mathcal{D}} \downarrow \alpha)$ is isomorph to $\partial(\overrightarrow{\Theta} \downarrow R)$, identifying the latching map $L_\alpha F_{P,Q} \to \text{よ}(R)$ with the canonical map $\partial R \to \text{よ}(R)$, which is a cofibration according to Lemma A.9. Hence, using Theorem 19.9.1.(2) from [11], we have a weak equivalence

$$\mathsf{hocolim}_{\mathcal{D}} F_{P,Q} \xrightarrow{\sim} \mathsf{colim}_{\mathcal{D}} F_{P,Q} \cong \text{よ}(P) \times \text{よ}(Q) \quad .$$

Then using Theorem 19.4.4.(1) in *loc.cit.* (we may because every object is cofibrant), we have a weak equivalence

$$\mathsf{Map}(\text{よ}(P) \times \text{よ}(Q), X) \simeq \mathsf{holim}_{\mathcal{D}^{\mathrm{op}}} \mathsf{Map}(F_{P,Q}, X) \quad .$$

Moreover, there is for each $\alpha = (P \leftarrow R \to Q)$ a weak equivalence

$$\mathsf{Map}(\mathbb{1}, X) \cong X_* \simeq X_R \cong \mathsf{Map}(F_{P,Q}(\alpha), X)$$

between fibrant objects (recall that X_* and X_R are fibrant by Proposition 18.5.3.(2) in [11]). Hence using Theorem 18.5.3.(2) in *loc.cit.*, there is a weak equivalence

$$\mathsf{holim}_{\mathcal{D}^{\mathrm{op}}} \mathsf{Map}(\mathbb{1}, X) \simeq \mathsf{holim}_{\mathcal{D}^{\mathrm{op}}} \mathsf{Map}(F_{P,Q}, X)$$

By the previous reasoning, the righthand object is weakly equivalent to $\mathsf{Map}(\text{よ}(P) \times \text{よ}(Q), X)$. Similarly, we have $\mathsf{holim}_{\mathcal{D}^{\mathrm{op}}} \mathsf{Map}(\mathbb{1}, X) \simeq \mathsf{Map}(\mathsf{hocolim}_{\mathcal{D}} \mathbb{1}, X)$ by Theorem 19.4.4.(1) in *loc.cit.*. Finally, by definiton of the injective structure on $\mathsf{sPsh}(\Theta)$, and by Propositions 9.3.1.(1) and 9.3.2.(1) in *loc.cit.*, the pair of adjoint functors

$$\hat{\Delta} \underset{\mathsf{ev}_*}{\overset{\mathsf{const}}{\rightleftarrows}} \mathsf{sPsh}(\Theta) \qquad (\perp)$$

is a QUILLEN pair, so const preserve homotopy colimits (by Theorem 19.4.5.(1) in *loc.cit.*). So in particular (using Proposition 18.1.6 in *loc.cit.*) $\mathsf{hocolim}_{\mathcal{D}} \mathbb{1} \simeq \mathsf{const}(\mathsf{N}\mathcal{D}^{\mathrm{op}}) \simeq \mathbb{1}$ because $\mathsf{N}\mathcal{D}$ in contractible by Lemma A.10. Whence the weak equivalence $X_* \to \mathsf{Map}(\text{よ}(P) \times \text{よ}(Q), X)$. □

Postulate 3

A.12 The third postulate reflect the fact that weak equivalences are the objectwise weak equivalences in the injective model structure on $\mathsf{sPsh}(\Theta)$.

Postulate 4 and Postulate 5

A.13 Suspension functor in $\mathsf{sPsh}(\Theta)$. Let $\Theta_{\bullet\bullet}$ be the category of bipointed pasting schemes and points-preserving maps, and similarly let $\mathsf{sPsh}(\Theta)_{\bullet\bullet}$ denotes the simplicial category of bipointed simplicial presheaves over Θ (it is cocomplete, and even a model category according to Proposition 1.1.8 in [12]). Following Paragraph 4.2.1.11. in [17], we may construct a suspension functor $\$' : \mathsf{sPsh}(\Theta) \to \mathsf{sPsh}(\Theta)_{\bullet\bullet}$ by KAN-extending the suspension operation $\text{よ} \circ \$: \Theta \to \Theta_{\bullet\bullet} \hookrightarrow \mathsf{sPsh}(\Theta)_{\bullet\bullet}$. By construction, this extension restrict (up to a canonical isomorphism) to $よ \circ \$$ on the subcategory $\Theta \hookrightarrow \mathsf{sPsh}(\Theta)$, which yields the intertwining map.

A.14 Hom**-functor as a right adjoint of** $\$'$**.** Still following [17], we oberve that this KAN-extension admits a right adjoint, which sends the bipointed presheaf (X, a, b) to $\mathsf{Hom}_X(a, b)$ defined by

$$\mathsf{Hom}_X(x_0, x_1)_P = \mathsf{Map}_{\bullet\bullet}(\$'よ(P), (X, x_0, x_1))$$

where $\mathsf{Map}_{\bullet\bullet}$ denotes the simplicial mapping space of bipointed maps.

A.15 Quillen adjunction and pushout preservation. We make the further observation that Hom sends a map of bipointed fibrant presheaves $f : (X, x_0, x_1) \to (Y, y_0, y_1)$ to the map $\mathsf{Hom}_X(x_0, x_1) \to \mathsf{Hom}_Y(y_0, y_1)$ defined objectwise by the postcomposition

$$f_* : \mathsf{Map}_{\bullet\bullet}(\$'よ(P), (X, x_0, x_1)) \to \mathsf{Map}_{\bullet\bullet}(\$'よ(P), (X, x_0, x_1))$$

Which is a fibration (resp. trivial fibration) when f is a fibration (resp. trivial fibration), according to Proposition 9.3.1.(2) (resp. 9.3.2.(2)) in [11]. Hence, the adjunct pair $(\$' \dashv \mathsf{Hom})$ is a QUILLEN pair according to Proposition 8.5.3 in *loc.cit.* Such a QUILLEN adjunction should yield the right notion of adjunction up to homotopy, and also ensures that $\$' : \mathsf{sPsh}(\Theta) \to \mathsf{sPsh}(\Theta)_{\bullet\bullet}$ preserve homotopy colimits.

Postulate 6

A.16 If $[\![X]\!]$ is a (fibrant) object of $\mathsf{sPsh}(\Theta)$ interpreting a (crisp) type $X :: \mathcal{U}$, then the interpretation of X_P for some $P :: \mathsf{PS}$ is given by $[\![X_P]\!] = \mathsf{Map}(よ(P), [\![X]\!]) \cong [\![X]\!]_P$. Howether, because sums are computed objectwise, the P-cells of a sum are canonically equivalent to the sums of the P-cells (see for instance Paragraph 5.1.2 in [19]). Which justify the postulated equivalence.

Postulate 7

A.17 Truncations are computed objectwise. Let X be a fibrant object in $\mathsf{sPsh}(\Theta)$, by definition (see 5.5.6.1 in [19]), X is n-truncated iff all the mapping spaces into it are n-truncated as KAN complexes. Note that this implies that each $X_P \simeq \mathsf{Map}(よ(P), X)$ is n-truncated. And conversely, if each X_P is n-truncated, then because every other object Y in $\mathsf{sPsh}(\Theta)$ is a colimit of representable, the mapping space $\mathsf{Map}(Y, X)$ is a limit of n-truncated KAN complexes, so is n-truncated by Proposition 5.5.6.5 in *loc.cit.* As a consequence, truncations may be computed objectwise.

A.18 Effective epimorphisms in $\mathsf{sPsh}(\Theta)$**.** According to Proposition 7.2.1.14 in [19], an effective epimorphism in $\mathsf{sPsh}(\Theta)$ is the same as an effective epimorphism in its underlying 1-topos, which is given by its homotopy category according to 5.5.6.2 in *loc.cit.*, that is $\hat{\Theta}$, where the truncation is computed objectwise. Moreover, we know that effective epimorphisms coincide with epimorphisms in a 1-topos, and more precisely to objectwise surjections in the case of $\hat{\Theta}$.

A.19 Semantic of Coverage. Recall that PS is interpreted as a constant, set valued presheaf. So if X is the presheaf modeling the type A of Postulate 7, then the sum $\sum_{P::PS} \sum_{c:A_P} よP$ will be interpreted as the coproduct

$$\coprod_{P \in \mathcal{O}b(\Theta)} X_P \times よ(P)$$

Then the map $\coprod_{P \in \mathcal{O}b(\Theta)} X_P \times よ(P) \to X$ will be an effective epimorphism iff all maps

$$\coprod_{P \in \mathcal{O}b(\Theta)} \pi_0(X_P) \times \mathsf{Hom}_\Theta(Q, P) \to \pi_0(X_Q) \qquad (Q \in \mathcal{O}b(\Theta))$$

are surjective. Indeed, this is implied by surjectivity of the Q-th component

$$\pi_0(X_Q) \times \mathsf{Hom}_\Theta(Q, Q) \to \pi_0(X_Q) \quad .$$

So this motivate our postulate of $\sum_{P:PS} \sum_{c:A_P} よP \to A$ being an effective epimorphism, assuming the type theoretic effective epimorphisms to indeed be modeled by effective epmorphisms in the higher categorical semantic.

References

[1] *nlab entry:* $(\infty, 1)$*-local geometric morphism*. https://ncatlab.org/nlab/show/%28%E2%88%9E%2C1%29-local+geometric+morphism.

[2] D. Ara, *Sur les infini-groupoïdes de Grothendieck et une variante infini-catégorique*, PhD thesis, 2010. Thèse de doctorat dirigée par Maltsiniotis, Georges Mathématiques Paris 7 2010.

[3] M. F. Atiyah, *Topological quantum field theory*, Publications Mathématiques de l'IHÉS, 68 (1988), pp. 175–186.

[4] J. C. Baez, *Quantum quandaries: a category-theoretic perspective*, 2004.

[5] C. Berger, *Iterated wreath product of the simplex category and iterated loop spaces*, Advances in Mathematics, 213 (2007), p. 230–270.

[6] J. E. Bergner and C. Rezk, *Reedy categories and the* Θ*-construction*, 2012.

[7] U. Buchholtz, F. van Doorn, and E. Rijke, *Higher groups in homotopy type theory*, 2018.

[8] D.-C. Cisinski and G. Maltsiniotis, *La catégorie* Θ *de joyal est une catégorie test*, Journal of Pure and Applied Algebra, 215 (2011), pp. 962–982.

[9] L. Fajstrup, E. Goubault, E. Haucourt, S. Mimram, and M. Raussen, *Directed algebraic topology and concurrency*, vol. 138, Springer, 2016.

[10] A. Grothendieck, *Pursuing stacks*, 1984.

[11] P. S. Hirschhorn, *Model categories and their localizations*, American Mathematical Society, 2003.

[12] M. Hovey, *Model Categories*, American Mathematical Society, 1999.

[13] P. T. Johnstone and I. Moerdijk, *Local maps of toposes*, Proceedings of the London Mathematical Society, s3-58 (1989), pp. 281–305.

[14] A. Joyal, *Disks, duality and* Θ*-categories*, 1997. Unpublished note: https://ncatlab.org/nlab/files/JoyalThetaCategories.pdf.

[15] D. M. Kan, *Abstract homotopy*, Proceedings of the National Academy of Sciences, 41 (1955), pp. 1092–1096.

[16] ——, *A combinatorial definition of homotopy groups*, Annals of Mathematics, 67 (1958), pp. 282–312.

[17] F. Loubaton, *Theory and models of* (∞, ω)*-categories*, 2023.

[18] P. L. Lumsdaine and M. Shulman, *Semantics of higher inductive types*, Mathematical Proceedings of the Cambridge Philosophical Society, 169 (2019), p. 159–208.

[19] J. Lurie, *Higher topos theory*, 2008.

[20] P. R. North, *Towards a directed homotopy type theory*, Electronic Notes in Theoretical Computer Science, 347 (2019), pp. 223–239.

[21] P. W. O'Hearn and D. J. Pym, *The logic of bunched implications*, Bulletin of Symbolic Logic, 5 (1999), p. 215–244.

[22] H. Perry and H. Kuen-Bang, *A note on left adjoints preserving colimits in hott*, 2025. unpublished work: https://hott-uf.github.io/2025/abstracts/HoTTUF_2025_paper_9.pdf.

[23] T. Porter, *Letter to grothendieck*, 1983. https://ncatlab.org/nlab/files/16-06-1983.pdf.

[24] C. Rezk, *A cartesian presentation of weak n-categories*, Geometry and Topology, 14 (2010), p. 521–571.

[25] E. Riehl and M. Shulman, *A type theory for synthetic* ∞*-categories*, 2023.

[26] M. Riley, *A bunched homotopy type theory for synthetic stable homotopy theory*, 2022. Ph.D. Thesis.

[27] U. SHOËPP, *Names and binding in type theory*, 2003. Ph.D. Thesis.

[28] M. SHULMAN, *The univalence axiom for elegant reedy presheaves*, 2015.

[29] ——, *Brouwer's fixed-point theorem in real-cohesive homotopy type theory*, 2017.

[30] THE UNIVALENT FOUNDATIONS PROGRAM, *Homotopy Type Theory: Univalent Foundations of Mathematics*, https://homotopytypetheory.org/book, Institute for Advanced Study, 2013.

[31] B. VAN DEN BERG AND R. GARNER, *Types are weak ω-groupoids*, Proceedings of the London Mathematical Society, 102 (2010), p. 370–394.

[32] V. VOEVODSKY, *Univalent foundations project, nsf grant application*, 2010.