A cellular type theory

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Abstract

Homotopy type theory comes equipped with a canonical semantics in which types are interpreted as ∞-groupoids. Over the recent years, a series of works have started extending this approach in order to reach a setting in which types can be more generally interpreted as higher categories. In particular, Riehl and Shulman have introduced the simplicial type theory which can be modeled in simplicial spaces, where $(\infty, 1)$-categories can be characterized as those satisfying two properties, namely Segalness and completeness. Here, we follow a similar path, and introduce a type theory with models in cellular spaces (space-valued presheaves over the category Θ), where we have a notion of (∞, ω)-category by imposing two similar conditions, following the ideas of Joyal, Rezk and Berger. More precisely, our type theory postulates a category of pasting schemes (which formally axiomatizes the category Θ) and a Yoneda embedding (which ensures that types behave as cellular spaces). Axiomatizing this requires us to be able to consider the underlying spaces of cellular spaces: this is made possible thanks to the presence of a comodality $\flat$, reminiscent of Shulman's crisp type theory. We introduce a notion of (∞, ω)-category in this setting and illustrate the applicability of our approach by showing various results on those, such as the fact that they are stable under taking sums, finite limits or homs, or that representable types are categories. We also provide, in appendix, a semantics justifying the pertinence of our axioms.

CCS Concepts

• **Theory of computation → Constructive mathematics**; **Type theory**.

1 Introduction

1.1 Homotopy and type theory

The investigation of Martin-Löf's intentional type theory [44] has revealed that identity types can bear non-trivial information, in the sense that two proofs of equality are not necessarily themselves equal. This observation was first formalized by Hofmann and Streicher [30] by constructing a model, where contexts are interpreted as groupoids, which does not validate the principle of uniqueness of identity proofs. Later on, Voevodsky and collaborators have introduced homotopy type theory [59], based on a new axiom, called *univalence*, validated in a model where contexts are interpreted as spaces (impersonated by simplicial sets) up to homotopy [33]: this followed pioneering work interpreting dependent type theory with identity types in model categories [4, 21], and formalizes the intuition that a type corresponds to a space, a term to a point in this space, a proof of equality to a path, a proof of equality between equalities as a homotopy between paths, and so on. This later model generalizes the groupoid model, in the sense that, under the Grothendieck hypothesis [27], spaces correspond to ∞-*groupoids*, a variant of the notion of groupoid comprising higher cells and where all structural axioms only hold up to higher coherence cells, which should themselves satisfy coherence laws up. More precisely, types in Voevodsky's model are interpreted as Kan complexes, which can be taken as a definition for ∞-groupoids in the simplicial setting. Moreover, it was observed early on that identities equip types in intentional type theory with a structure of ∞-groupoid [12, 40, 60] in the sense of Grothendieck-Batanin-Leinster [6, 27, 36].

The type theory obtained by adding the univalence axiom is called *homotopy type theory* because it allows to reason in a synthetic way on homotopy types. Formalizing geometric constructions in this setting is interesting for multiple reasons: the resulting proofs can be fully detailed and checked in proof assistants such as Agda or Rocq, all manipulations performed there are invariant up to homotopy by construction, and they automatically generalize to all the models of the type theory such as those which can be found in ∞-toposes [58]. Important results have been now formalized in this setting such as the computation of the 4-th homotopy group of the 2-sphere [12, 38], the Blakers-Massey theorem [2], the stabilization of higher groups [13], and so on.

1.2 Directed homotopy type theory

More generally than modeling ∞-groupoids, one would like to have a variant of type theory where types are (∞, n)-categories, i.e. weak categories in which morphisms are only invertible starting from dimension n, with n possibly being ω, in which case we do not require any morphism to be invertible. These structures appear naturally in category theory: for instance, the collection of ∞-groupoids forms an $(\infty, 1)$-category. On the topological side, $(\infty, 1)$-categories arise as fundamental categories of *directed spaces* [20], which are variants of the notion of fundamental group adapted to a setting where topological space are equipped with a notion of "time direction" which allows identifying, among paths, the ones that should be considered as being properly directed.

From this perspective has emerged the hope for a *directed type theory*, which would allow to synthetically consider higher categories. Early on, it was observed that one cannot expect such a theory to be defined as a simple variant of dependent type theory. One reason is that models of such a theory would have to be locally cartesian closed categories, a property which is not satisfied by the category of 1-categories (exponentiable functors can be characterized as those being Conduché fibrations [19, 22]), nor by higher categories (for similar reasons). This means the naive hope that one could come up with a variant of the rules for identity types which would allow for taking in account directed morphisms is doomed to fail. This however does not prevent one from crafting more creative type theories with dedicated construction in order to handle directed morphisms, and various investigations have been made in this direction. A type theory which can be interpreted in (strict) 2-categories is proposed in [37], and a variant adapted to bicategories is introduced in [1]: the 1-cells there correspond to reductions of terms, and the syntax internalizes the operations and axioms expected to be satisfied in the bicategorical semantics. A type theory closer to traditional intentional type theory and featuring a form of transport is proposed in [45], where the type theory is extended with a "core" modality as well as a dualizing operation, and interpreted in the category of small categories, see also [46, 62] for related approaches.

More recently, Riehl and Shulman [52] have introduced *simplicial type theory*, which is a type theory adapted to $(\infty, 1)$-categories, whose starting point is the introduction of a type corresponding to a directed interval. The manipulations there are not entirely synthetic, in the sense that types are not always to be interpreted as $(\infty, 1)$-categories, but rather as simplicial spaces, among which we can identify categories as *complete Segal types*, which can roughly be described as types supporting composition and in which paths correspond to equivalences. This means that, when defining operations on categories, one should always make sure that the result actually is a category (as opposed to homotopy type theory where we can implicitly suppose that all operations preserve being an ∞-groupoid). Nevertheless, many recent developments using this type theory have shown its applicability for reasoning in a concise and formal way about $(\infty, 1)$-categories [14, 25, 26, 63, 64]. We note that, in order to make the type theory useful in practice, it is often extended with a modality $\flat$ (and possibly more modalities, based on crisp type theory [57] or the multimodal extension of intentional type theory [24]) which can semantically be interpreted as taking the *core* of a type, i.e. keeping only its weakly invertible morphisms. This kind of approach in homotopy type theory originates in the work of Shulman who introduced *crisp type theory* [57], and is also expected to play an important role in the current axiomatization of higher categories by Cisinski and collaborators [17].

1.3 A fully directed type theory

This work constitutes a first step toward the generalization of the previous work toward a type theory which is "fully directed", in the sense that our type theory contains types which can be identified to (∞, ω)-categories, i.e. weak higher categories, where no cell is supposed to be invertible. Semantically, our starting point compared to simplicial type theory consists in replacing simplicial spaces by Θ-spaces. We recall that the category Θ, due to Joyal [32], is the category whose objects are pasting schemes (i.e. formal composites of globes) and morphisms are functors between them. One can then consider *cellular spaces*, which are presheaves enriched in spaces over Θ, and isolate Θ-*spaces* as being cellular spaces A which satisfy a Segal-type condition (given a pasting scheme P, the canonical map from the space A_P to the canonical fibered product of spaces A_{P_i} should be a weak equivalence) and a completeness condition (paths in spaces correspond to equivalences). These were introduced by Rezk as a model for (∞, ω)-categories [50, 51]. In the original definition, the notion of "space" there is axiomatized by simplicial sets equipped with the Quillen model structure. In fact, by restricting to n-dimensional pasting schemes, one obtains a notion of (∞, n)-category which, for $n = 1$, coincides with Segal spaces (in particular Θ_1 is the simplicial category Δ). It is thus tempting to try to generalize simplicial type theory in order to accommodate for cellular spaces: this is precisely the objective of the present work. This generalization is not immediate: simplicial type theory is based on a layer of *topes* which allows considering subshapes of cubes in which one can encode the simplicial machinery, but there is no obvious generalization at our disposal in order to encode the category Θ, which must therefore be added axiomatically. In particular, simplicial type theory features a directed interval type $\mathbb{I}$, so that 1-cells in a type A correspond to maps $\mathbb{I} \to A$, 2-cells to maps $\mathbb{I}^2 \to A$ and so on. This approach would not work in the fully directed case: all the m-cells of $\mathbb{I}^n$ are reversible for $m > 1$ and $n > 1$, and thus maps $\mathbb{I}^n \to A$ only detect reversible n-cells in A.

1.4 Cellular type theory

Our type theory results from several successive extensions of homotopy type theory. We first formally add a type corresponding to pasting schemes in the type theory, which is made possible thanks to the nice combinatorial structure governing the category Θ. In a second time, interpreting all our types as cellular presheaves, we axiomatize the Yoneda embedding of Θ into types, which should give us access to the spaces of cells in the types by the Yoneda lemma. In order for this to be possible, we need to have access to a notion of space, whereas our types are meant to be cellular spaces: this motivates the introduction of a modality $\flat$ which restricts a cellular space to the underlying space, following the principles of crisp type theory [57]. We claim that the resulting type theoretic framework is suitable for working synthetically with cellular spaces: in order to support this, we define weak higher categories and perform several constructions with them.

1.5 Plan of the paper

We begin by recalling categorical definitions around the Θ category and cellular spaces (Section 2), we then introduce our type theoretic setting obtained by adding axioms to intuitionistic type theory (Section 3), and finally we define and study higher categories (Section 4): in particular, we show that discrete types are categories (Section 4.3), that the homotopy level of a type is determined pointwise (Section 4.4), that categories are closed under taking homs (Section 4.5), that representable types are categories (Section 4.6) and that categories are stable under finite limits and coproducts (Section 4.7).

All the proofs omitted in the article can be found in Section A. The semantics of our type theory is detailed in Section B and we provide extensions about codiscrete types (Section C) and (∞, n)-categories (Section D).

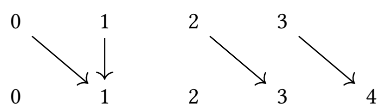
2 The category Θ

We recall the definition of the category Θ due to Joyal [32], see also [8, 18, 43]. Many of the results in this section have been formalized in [35].

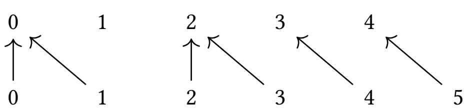
2.1 The simplicial and interval categories

We first need to fix some notations and recall basic constructions. Given a natural number n, we write $[n]$ for the (totally ordered) set $\{0, 1, \ldots, n\}$. The *simplicial category* Δ is the category whose objects are the natural numbers and where a morphism $f : m \to n$ is a non-decreasing function $[m] \to [n]$. A *simplicial set* is a presheaf over this category. We recall that there is a standard model structure on the category $\hat{\Delta}$ of simplicial sets, often called the Kan-Quillen model structure, whose weak equivalences are the morphisms inducing weak equivalences of topological spaces on the geometric realizations and whose cofibrations are the monomorphisms [23]. Trivial fibrations are generated by horn inclusions, so that fibrant objects are precisely Kan complexes.

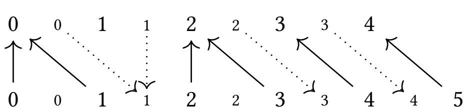
The *category of intervals* $\mathcal{I}$ is the category whose objects are natural numbers and morphisms $f : m \to n$ are non-decreasing functions $[m+1] \to [n+1]$ which preserve the minimal and the maximal elements. There is an isomorphism $\Delta \xrightarrow{\sim} \mathcal{I}^{\mathrm{op}}$ which sends an object n to n (i.e. $[n]$ to $[n+1]$) and sends a map $f : [m] \to [n]$ to the map $f^\vee : [n+1] \to [m+1]$ with is defined on $i \in [n+1]$ by $f^\vee(i) = \min\{j \in [m+1] \mid f(j) \geq i\}$, see [32] for details. For instance, consider the following map $3 \to 4$ in Δ:


(1)

It corresponds to the following map in $\mathcal{I}^{\mathrm{op}}$:


(2)

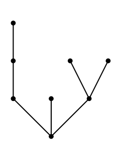
Intuitively, one can see an element i in (2) as corresponding to the interval between the elements $(i-1)$ and i in (1). We can indeed superimpose the map (1) (in dotted) with the map (2) as follows:



Moreover, the minimal and maximal elements correspond to the semi-infinite interval on the left and on the right respectively, which explains intuitively why we will consider them as "undefined" values in the definition of morphisms in Θ below.

2.2 Pasting schemes

We define the set of *pasting schemes* as the smallest set closed under formal products of arbitrary finite arity: this means that for every natural number n and pasting schemes $P_1, \ldots, P_n$, we have a pasting scheme $[P_1, \ldots, P_n]$ corresponding to their formal product. More formally, pasting schemes are the initial algebra of the polynomial functor $X \mapsto \bigsqcup_{n \in \mathbb{N}} X^n$ on **Set**. Alternatively, pasting schemes can also be thought of as finite planar trees, sometimes called Batanin trees [6, 7]. For instance, the tree corresponding to the pasting scheme $[[[[]]], [], [[], []]]$ is pictured on the right. Note that this tree is essentially the syntax tree of the expression corresponding to the pasting scheme, which should help understanding the correspondence between the two representations.



In the following, given a natural number n, we write O_n for the *n-disk* pasting scheme $[[\cdots[]\cdots]]$ obtained by applying n times $[-]$ to $[]$, and we write $[n]$ for the pasting scheme $[[], [], \ldots, []]$, with n copies of $[]$. The *dimension* $\dim(P)$ of a pasting scheme P is the depth of the corresponding planar tree: it can be defined inductively by $\dim([P_0, \ldots, P_n]) = 1 + \max_i \dim(P_i)$ with the convention $\dim([]) = 0$.

2.3 Globular sets

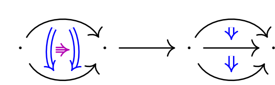
We write $\mathcal{G}$ for the category whose objects are natural numbers and morphisms are generated by $s_n, t_n : n \to n+1$ for $n \in \mathbb{N}$, respectively called *source* and *target* maps, subject to the relations $s_{n+1} \circ s_n = t_{n+1} \circ s_n$ and $s_{n+1} \circ t_n = t_{n+1} \circ t_n$ for $n \in \mathbb{N}$. We write $\hat{\mathcal{G}}$ for the category of presheaves over $\mathcal{G}$, whose objects are also known as *globular sets*. Given $G \in \hat{\mathcal{G}}$ and an object $n \in \mathcal{G}$, we write G_n for the set obtained as the image of n under G, whose elements are called *n-cells*, and $s_n^G, t_n^G : G_{n+1} \to G_n$ for the respective images of the morphisms s_n and t_n. We write 1 for the the globular set with one 0-cell $\star$ and no cell of higher dimension (this is not the terminal one).

A bipointed globular set is a globular set G equipped with two distinguished 0-cells, or equivalently with two maps left, right : $1 \to G$. We write $\hat{\mathcal{G}}_{..}$ for the corresponding category, with maps preserving distinguished elements. The forgetful functor $\hat{\mathcal{G}}_{..} \to \hat{\mathcal{G}}$ admits a left adjoint $\mathrm{S} : \hat{\mathcal{G}} \to \hat{\mathcal{G}}_{..}$, called *suspension*, sending a globular set G to the globular set SG defined by $(SG)_0 \equiv \{-, +\}$ (with $-$ and $+$ as distinguished 0-cells) and $(SG)_{n+1} \equiv G_n$ with source and target maps given by $s_0^{SG}(x) \equiv -$, $t_0^{SG}(x) \equiv +$, $s_{n+1}^{SG} \equiv s_n^G$ and $t_{n+1}^{SG} \equiv t_n^G$.

Any pasting scheme induces a globular set $P^{\mathcal{G}}$ defined inductively by $[]^{\mathcal{G}} \equiv 1$ and $[P_1, \ldots, P_n]^{\mathcal{G}}$ is the colimit of the diagram

$$SP_1^* \xleftarrow{\iota^+} 1 \xrightarrow{\iota^-} SP_2^* \xleftarrow{\iota^+} 1 \xrightarrow{\iota^-} \ldots \xleftarrow{\iota^+} 1 \xrightarrow{\iota^-} SP_n^*$$

For instance, we have

$[[[[]]], [], [[], []]]^{\mathcal{G}} \quad =$ 

We write $\mathbf{Cat}_\omega$ for the category of (strict) ω-categories, which are globular sets equipped with identities and compositions satisfying suitable axioms. By general results about locally presentable categories [5, Theorem 3.5], the forgetful functor $\mathbf{Cat}_\omega \to \hat{\mathcal{G}}$ admits a left adjoint, constructing the free ω-category on a globular set. Given a pasting scheme P, we write P^* for the free ω-category on the globular set $P^{\mathcal{G}}$.

2.4 The category Θ

The category Θ is defined as the full subcategory of $\mathbf{Cat}_\omega$ whose objects are of the form P^* for some pasting scheme P [32]. The operation $-^*$ on pasting schemes is injective [7, 8] so that we may safely refer to an object of Θ as a pasting scheme. Given a natural number n, the category Θ_n is obtained by restricting Θ to pasting schemes of dimension at most n, so that Θ can be recovered as the inductive limit of the Θ_n. In particular, for $n = 1$, the category Θ_1 coincides with the *simplicial category* Δ.

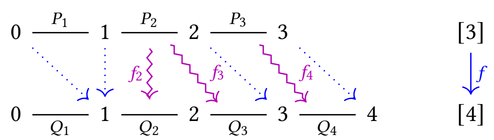
The category Θ_{n+1} can be recovered from Θ_n as the wreath product $\Delta \wr \Theta_n$, see [8]. This means that we have the following inductive description of the morphisms of Θ:

Definition 1. The category Θ is the category where an object is a pasting scheme and a morphism $f : [P_1, \ldots, P_n] \to [Q_1, \ldots, Q_n]$ consists of

- a map $f : m \to n$ in Δ,
- for every $i \in [n+1]$ such that $0 < f^\vee(i) < m+1$, a map $f_i : P_{f^\vee(i)} \to Q_i$.

Identities and composition are induced by those in Δ and Θ (recursively).

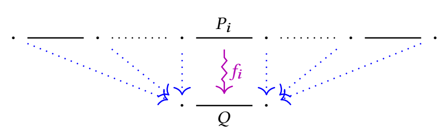
For instance, we have $f : [P_1, P_2, P_3] \to [Q_1, Q_2, Q_3, Q_4]$ induced by the function f of (1) and the morphisms $f_2 : P_2 \to Q_2$, $f_3 : P_2 \to Q_3$ and $f_4 : P_3 \to Q_4$. This can be pictured as follows:



The above description should convince the reader that we can implement data structures in order to describe objects and morphisms of the category Θ. More precisely, both the objects and the morphisms of this category can be described as inductive types.

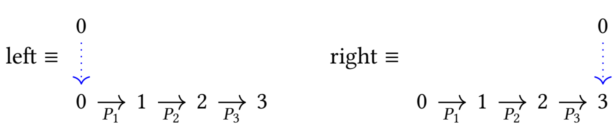
A morphism $f : P \to Q$ is *bipointed* when P and Q are both different from [] and the underlying function in Δ preserves the endpoints, i.e. belongs to $\mathcal{I}$. We write $\Theta_{..}$ for the subcategory of Θ with all pasting schemes excepting [] as objects and bipointed morphisms. We write $\sigma_i : \mathbb{N} \to \{0, 1\}$ for the "step" function such that $\sigma_i(j) = 0$ for $j < i$ and $\sigma_i(j) = 1$ otherwise. The following lemma follows immediately from the definition of $\Theta_{..}$:

Lemma 2. *Given pasting schemes $P \equiv [P_1, \ldots, P_n]$ and Q, the morphisms $f : [P_1, \ldots, P_n] \to [Q]$ which are bipointed are of the form $(\sigma_i, (f_i))$ for some i with $0 < i \le n$ and morphism $f_i : P_i \to Q$. Graphically,*



2.5 Operations in Θ

2.5.1 Source and target. Given a pasting scheme P there are two canonical maps left, right : $[] \to P$ whose underlying map in Δ is the map $[0] \to [n]$ respectively sending 0 to 0 and n. For instance, for $P \equiv [P_1, P_2, P_3]$ those maps can respectively be pictured as

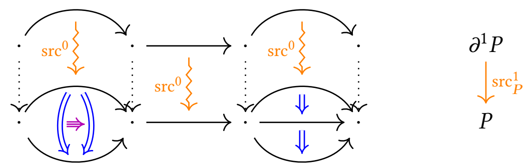


Thanks to these maps, any pasting scheme can canonically be considered as being bipointed.

Given $k \in \mathbb{N}$, any pasting scheme P induces a pasting scheme $\partial^k P$, its k-dimensional *boundary*, along with boundary morphisms $\mathrm{src}^k_P, \mathrm{tgt}^k_P : \partial^k P \to P$, the *source* and *target* maps, defined by induction on k as follows. In the base case, we define $\partial^0 P \equiv []$ with $\mathrm{src}^0_P \equiv$ left and $\mathrm{tgt}^0_P \equiv$ right. In the inductive case, we define $\partial^{k+1}[P_1, \ldots, P_n] \equiv [\partial^k P_1, \ldots, \partial^k P_n]$ with

$$\mathrm{src}^{k+1}_{[P_1,\ldots,P_n]} = (\mathrm{id}_n, (\mathrm{src}^k_{P_i})_{1\le i\le n}) \qquad \mathrm{tgt}^{k+1}_{[P_1,\ldots,P_n]} = (\mathrm{id}_n, (\mathrm{tgt}^k_{P_i})_{1\le i\le n})$$

We simply write ∂P for $\partial^{\dim P - 1}$, when $\dim P > 0$, and similarly for src_P and tgt_P. For instance, for $P = [[[[]]], [], [[], []]]$, the map $\mathrm{src}^1_P : \partial^1 P \to P$ corresponds to the inclusion of globular sets depicted below:



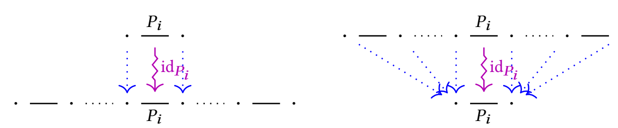
It can be observed that any pasting scheme is "acyclic", in the sense that if it contains two cells in opposite directions then those are equal.

Lemma 3. *Suppose given a pasting scheme P together with two morphisms $a, b : \mathrm{O}_{n+1} \to_{\mathrm{PS}} P$ such that $a \circ \mathrm{src} = b \circ \mathrm{tgt}$ and $b \circ \mathrm{src} = a \circ \mathrm{tgt}$. Then $a \circ \mathrm{src} = a \circ \mathrm{tgt}$ and $a = b$.*

2.5.2 Inclusion and retraction. Suppose given a pasting scheme $P \equiv [P_1, \ldots, P_n]$. We have, for any index i, a pair of morphisms $\subseteq_i : [P_i] \to P$ and $\pi_i : P \to [P_i]$ forming a section-retraction pair, i.e. $\pi_i \circ \subseteq_i = \mathrm{id}_{[P_i]}$. Those are defined by

$$\subseteq_i \equiv (j \mapsto i + j - 1, (\mathrm{id}_{P_i})) \qquad \pi_i \equiv (\sigma_i, (\mathrm{id}_{P_i}))$$

and can respectively be illustrated as follows:



2.5.3 Suspension. The operation which to a pasting scheme P associates $[P]$ extends as a functor $\mathrm{S} : \Theta \to \Theta_{..}$, called *suspension*. It is defined on objects by $\mathrm{S}P \equiv [P]$ and, given a morphism $f : P \to Q$, the morphism $\mathrm{S}f : [P] \to [Q]$ is the function consisting of the identity $\mathrm{id} : 1 \to 1$ in Δ and the morphism f.

2.5.4 Wedge sum. Given two pasting schemes $P \equiv [P_1, \ldots, P_n]$ and $Q \equiv [Q_1, \ldots, Q_m]$ their *wedge sum* $P \vee Q$ is the pasting scheme $[P_1, \ldots, P_n, Q_1, \ldots, Q_m]$ corresponding to the concatenation of lists.

2.6 Cellular spaces and categories

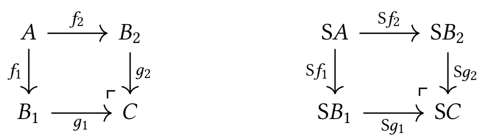
2.6.1 Cellular spaces. A *cellular space* X is a simplicial presheaf over the category Θ, i.e. a map $\Theta \to \hat{\Delta}$, which is fibrant with respect to the injective model structure, where $\hat{\Delta}$ is equipped with the Kan-Quillen model structure. Given $P \in \Theta$, we write X_P for the image of P under X: an object of X_P is sometimes called a *P-cell* of X. The category Θ can be shown to be an elegant Reedy category so that the injective and Reedy model structures on it coincide [9].

Alternatively, the Kan-Quillen model structure on simplicial sets can be thought of as presenting the ∞-category $\mathcal{S}$ of *spaces*, whose objects are ∞-groupoids, impersonated here by Kan complexes [42, Section 1.2.16]. With this perspective, the cellular spaces are the objects of the ∞-category $\mathrm{Psh}(\Theta) \equiv \mathrm{Fun}(\Theta^{\mathrm{op}}, \mathcal{S})$ of *spatial presheaves* over Θ, see [42, Section 5.1] for details. As for any presheaf category, we have a Yoneda embedding $\text{よ} : \Theta \to \mathrm{Psh}\,\Theta$ [42, Section 5.1.3]. We can define a suspension operation on cellular spaces as the left Kan extension of the functor $\text{よ} \circ \mathrm{S}$ along よ, which we still write as $\mathrm{S} : \mathrm{Psh}(\Theta) \to \mathrm{Psh}(\Theta)$. This operation actually defines a functor from $\mathrm{Psh}(\Theta)$ to the category $\mathrm{Psh}(\Theta)_{..}$ of bipointed presheaves (equipped with two distinguished $[]$-cells), see [47, Section 4.2], which preserves colimits. The cells of the suspension can be characterized as follows:

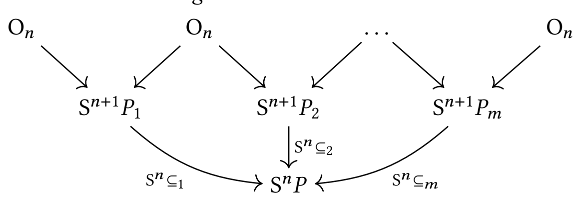
Proposition 4. *Given a cellular space X and a pasting scheme $P \equiv [P_1, \ldots, P_n]$, we have $(\mathrm{S}X)_P = 1 + \sum_i X_{P_i} + 1$.*

Simple combinatorics using the definitions of morphisms in Θ implies the following result, that will be needed in 4.6:

Lemma 5. *If the square on the left is a pushout in Θ, then so is the one on the right.*



Lemma 6. *Given a pasting scheme $P \equiv [P_1, \cdots, P_m]$, the following cocone in Θ is colimiting:*



Proof. By unfolding Definition 1, we see that the cocone for $n = 0$ is colimiting, and deduce the general result by using Lemma 5. ◻

2.6.2 *Θ-spaces.* A cellular space A is a Θ-*space* when moreover

- it satisfies the *Segal condition*: for every pasting schemes P and Q and $n \geq 0$, the canonical map

$$A_{\mathrm{S}^n(P \vee Q)} \to A_{\mathrm{S}^n P} \times_{A_k} A_{\mathrm{S}^n Q}$$

  is an equivalence,
- it satisfies the *completeness condition*: for every dimension n, the canonical map from n-cells to $(n+1)$-equivalences is a weak equivalence.

The Θ-spaces have been advocated as being a good notion of (∞, ω)-category [3, 10, 11, 50, 51]. They are the fibrant objects of a model structure on the category of simplicial presheaves over Θ [50]. An alternative definition of the notion of Θ-space, closer to the one that we use subsequently, can be found in [39, Section 4.2.1.6].

3 Type theoretic setting

We consider a type theory based on homotopy type theory, by which we mean intuitionistic type theory [44] together with the univalence axiom [59]. More precisely, we suppose that our theory features dependent sums, dependent products, identity types and a countable hierarchy of univalent universes. We also suppose that we have access to the usual basic data types (the terminal type, booleans, natural numbers, lists, etc.) as well as homotopy pushouts (this is in particular satisfied if we assume that we have all higher inductive types) so that we have all finite colimits. In addition to this now fairly standard material, we suppose that our type theory is spatial (in the sense that it features a modality $\flat$ as explained below) and that we have types corresponding to the objects and hom-sets of the category Θ. Although we cannot recall all the standard rules in details here, we at least need to fix some notations, before introducing our axioms.

3.1 Notations

Given $\ell \in \mathbb{N}$, we write $\mathcal{U}_\ell$ for the universe at level ℓ, or simply $\mathcal{U}$ when size issues can easily be handled. A *context* Γ is a finite list $x_1 : A_1, \ldots, x_n : A_n$ of pairs consisting of a variable and a type. We write $\Gamma \vdash t : A$ for the judgment indicating that t is a term of type A in the context Γ. In particular, a judgment of the form $\Gamma \vdash A : \mathcal{U}$ means that A is a type. Given a type A and a type family $B : A \to \mathcal{U}$, we write $\Sigma(x : A).B(x)$ (resp. $\Pi(x : A).B(x)$ or $(x : A) \to B(x)$) for *dependent sum* and *dependent product* types. Given two types A and B, we write $A \times B$ and $A \to B$ for their *product* and *arrow* types, which are particular non-dependent cases of the previous ones; we also write $A \sqcup B$ or $A + B$ for their coproduct. The *terminal* type is noted 1 and its canonical element is noted $\star$. Given two terms t and u of common type A, we write $t =_A u$ (or simply $t = u$) for the type of *identities* or *paths* between them. Among those, we distinguish *definitional equalities* which are denoted $t \equiv u$. A morphism $f : A \to B$ is called an *equivalence* when it has both a left and a right inverse, we write $\mathrm{isEquiv}(f)$ for the predicate indicating that f is an equivalence and we write $A \simeq B$ for the type of equivalences between A and B. The *univalence* axiom states that the canonical map $(A = B) \to (A \simeq B)$ is an equivalence for every types A and B.

A type A is *contractible* (resp. a *proposition*, resp. a *set*) when it is equivalent to 1 (resp. every identity type is contractible, resp. every identity type is a proposition). We write Prop (resp. Set) for the type of propositions (resp. sets). One can more generally define a notion of n-*type* so that previous types correspond to the cases where n is -2, -1 and 0. We write $\|-\|_n$ for the n-*truncation* operation, which formally turns a type into an n-type.

3.2 Informal semantics

In order to fix ideas, we provide here informally the intended semantics of our type theory in Θ-spaces, a detailed presentation being given in Section B. This is a particular case of Shulman's general construction of a model of univalent type theory in spatial presheaves over any elegant Reedy category [56]. This interpretation can be thought of as a generalization of the traditional presheaf semantics [29, Section 4], replacing presheaves (enriched in sets) by spatial presheaves (enriched in space). Indeed, presheafs over a fixed category is canonically equipped with a structure of category with families, which allows interpreting intuitionistic type theory, in a way that supports all expected constructions on types.

In our intended interpretation, a context Γ is interpreted as a presheaf $[\![\Gamma]\!] \in \mathrm{Psh}(\Theta)$. In particular, the empty context is interpreted as the terminal presheaf. A type in the empty context is also interpreted as such a presheaf and, more generally, a type A in a context Γ is interpreted as a fibration $[\![A]\!] \to [\![\Gamma]\!]$. A term t of type A in a context Γ consists of a section $[\![t]\!]$ of the previous fibration. The interpretation of the universe $\mathcal{U}$ (in the empty context) is the presheaf in $\mathrm{Psh}(\Theta)$ sending a pasting scheme P to the space of (appropriately small) fibrations over よP.

3.3 Spatial type theory

The presheaf semantics supports other important operations, which are not reflected (yet) in the syntax. The inclusion $1 \to \Theta$, sending the object of 1 to the pasting scheme [] (which is the object of objects) induces, by precomposition, a functor $r : \mathrm{Psh}(\Theta) \to \mathrm{Psh}(1)$ restricting a cellular space to its space of objects (the ∞-category $\mathrm{Psh}(1)$ is canonically isomorphic to $\mathcal{S}$). This functor admits a left adjoint b, sending a space X to the constant presheaf equal to X.

$$\mathcal{S} \underset{r}{\overset{b}{\rightleftarrows}} \mathrm{Psh}(\Theta) \qquad (b \dashv r)$$

These functors induce a comonad $\flat$ on $\mathrm{Psh}(\Theta)$ which will allow us to consider spaces (as opposed to cellular spaces), by embedding them into cellular spaces. For instance, for an arbitrary cellular space, we want to be able to have access to the space of cells of a given shape (which is not a cellular space in an interesting way).

The modality $\flat$ being comonadic, it cannot be expressed directly. This motivated the introduction of *spatial type theory* [57], which axiomatizes this modality: any type A induces a type $\flat A$. In order to do so, we have to consider contexts containing two kinds of variables: the *crisp* ones and the usual (or *cohesives*) ones. The crisp variables are more constrained, in the sense that they can only be substituted by terms of modal type (and their type can only depend on crisp variables). We write $\Delta \mid \Gamma$ for a context where the variables in Δ are the crisp ones. The associated inference rules for $\flat$ are

$$\frac{\Delta \mid \cdot \vdash A : \mathsf{Type}}{\Delta \mid \Gamma \vdash \flat A} \qquad \frac{\Delta \mid \cdot \vdash t : A}{\Delta \mid \Gamma \vdash t^\flat : \flat A}$$

$$\frac{\Delta \mid \Gamma, x : \flat A \vdash B : \mathsf{Type} \qquad \Delta \mid \Gamma \vdash t : \flat A \qquad \Delta, y :: A \mid \Gamma \vdash u : B[y^\flat / x]}{\Delta \mid \Gamma \vdash \mathsf{let}\ y^\flat = t\ \mathsf{in}\ u : B[t/x]}$$

as well as the expected β-reduction rule. We write $x :: A$ to indicate that x is supposed to be of type A in the crisp part of the context. The two first rules ensure that modal terms or variables can only depend on the crisp part of the context, and the last one is such that a crisp variable can only be substituted by a modal term. Given a type $A :: \mathcal{U}$, we write $-_\flat : \flat A \to A$ for the canonical map defined by $x_\flat \equiv (\mathsf{let}\ y^\flat = x\ \mathsf{in}\ y)$; we say that A is *discrete* when this map is an equivalence. The $\flat$ modality is functorial in the sense that every function $f :: A \to B$ induces a function $\flat f : \flat A \to \flat B$ defined by $\flat f\,x \equiv (\mathsf{let}\ y^\flat = x\ \mathsf{in}\ (f\,y)^\flat)$. In order to experiment with such a theory a *flat* extension of Agda is available [61]. From now on, we suppose that our type theory is a crisp extension of intuitionistic theory, as indicated above.

These modalities will be particularly useful when considering categories below. Given a type A, the type $\flat A$ corresponds to restricting to the objects of A, or considering the global sections. In particular, when A is a category, this will amount to restricting to the *core* of A, obtained by keeping only invertible morphisms. With this point of view in mind, crisp variables can only depend on types which behave like groupoids, for which we do not have to handle variance issues (see Section 5 for further discussion on this point).

3.4 Wild categories

In type theory, the structure resulting from the direct translation of the notion of category is called a *wild category* [15]: it consists of a type $O :: \mathcal{U}$ of morphisms, a family of types of morphisms $M :: O \times O \to \mathcal{U}$, compositions and identities, which are associative and unital. This notion is “wrong” in the sense that we lack the higher coherences, but still useful in the sense that it approximates the right notion of ∞-category (which is expected to require an infinite amount of coherence datum and thus be difficult to formulate). Note that, in the context of spatial type theory, all the data comprised in wild categories is always assumed to be crisp. Similarly, the direct translation of the traditional notion of functor is called a *wild functor* and the corresponding notion of adjunction between two functors is called a *wild adjunction*. We should recall that some of the expected properties for those do not go through without additional hypothesis. In particular, wild left adjoints do not preserve colimits in general unless we suppose that they satisfy an additional 2-coherence property [49]. Moreover, all the adjunctions considered here will be *crisp* adjunctions, see [57], by which we mean that we have functors $L :: A \to B$ and $R :: B \to A$ together with isomorphisms $\flat(L\,a \to b) \simeq \flat(a \to R\,b)$ natural in a and b.

3.5 Pasting schemes

As a last important construction built in in our type theory, we suppose that we have types encoding the category Θ, which we explained how to handle algorithmically in Section 2. In more details, this means the following. We first suppose given a type PS of pasting schemes. If our type theory features inductive types, we can define PS as an inductive type with one constructor of type $\mathsf{List}\ \mathsf{PS} \to \mathsf{PS}$, i.e. a pasting scheme is a list of pasting schemes. Otherwise, the type PS can be directly axiomatized as a new type with the expected associated rules:

$$\frac{}{\Delta \mid \Gamma \vdash \mathsf{PS}} \qquad \frac{\Delta \mid \Gamma \vdash L : \mathsf{List}\ \mathsf{PS}}{\Delta \mid \Gamma \vdash [L] : \mathsf{PS}}$$

$$\frac{\Delta \mid \Gamma, x : \mathsf{PS} \vdash A : \mathsf{Type} \qquad \Delta \mid \Gamma, L : \mathsf{List}\ \mathsf{PS} \vdash t : A[[L]/x] \qquad \Delta \mid \Gamma \vdash P : \mathsf{PS}}{\Delta \mid \Gamma \vdash \mathsf{let}\ [L] = P\ \mathsf{in}\ t : A[P/x]}$$

$$\frac{\Delta \mid \Gamma, x : \mathsf{PS} \vdash A : \mathsf{Type} \qquad \Delta \mid \Gamma, L : \mathsf{List}\ \mathsf{PS} \vdash t : A[[L]/x] \qquad \Delta \mid \Gamma \vdash L : \mathsf{List}\ \mathsf{PS}}{\Delta \mid \Gamma \vdash (\mathsf{let}\ [L] = [L]\ \mathsf{in}\ t) \equiv t : A[[L]/x]}$$

We also suppose that for every pasting schemes $P\,Q : \mathsf{PS}$, we have a type $P \to_{\mathsf{PS}} Q$ which corresponds to the hom type in Θ between P and Q: again, from the description of Definition 1, this can be axiomatized as an inductive type (or directly as a built-in type), see [35] for a formalization.

Postulate 1 (Pasting schemes). *We have a type* PS *of pasting schemes as well as types* $P \to_{\mathsf{PS}} Q$ *for* $P\,Q : \mathsf{PS}$.

By induction, we define functions $\mathrm{id} : (P : \mathsf{PS}) \to (P \to_{\mathsf{PS}} P)$ and

$$- \circ - : (P\,Q\,R : \mathsf{PS}) \to (Q \to_{\mathsf{PS}} R) \to (P \to_{\mathsf{PS}} Q) \to (P \to_{\mathsf{PS}} R)$$

which respectively compute identities and composition of morphisms in Θ, so that they from a category in the sense of homotopy type theory in [59, Definition 9.1.1].

Proposition 7. *The above data forms a category* Θ *with* PS *as set of objects,* $\to_{\mathrm{PS}}$ *as sets of morphisms, and* id *and* $\circ$ *as identities and composition.*

Proof. Given $P\,Q$: PS, the definition of the type $P \to_{\mathrm{PS}} Q$ as an inductive type implies that it has decidable equality and is thus a set by Hedberg's theorem [59, Theorem 7.2.5]. The fact that composition is associative and unital can be shown by induction (this follows from the fact that we have a category by Definition 1). By similar arguments as above, one can show that the type PS is a set and that the type of automorphisms of a pasting scheme is contractible (because Θ is a Reedy category [9]), from which follows that this is indeed a (univalent) category, in the sense that it is complete, i.e. the canonical map from identities of objects to isomorphisms is an equivalence. ◻

3.6 Yoneda embedding

Since all our types are to be interpreted as cellular sets, and we now have access to the category Θ, it is natural to axiomatize the Yoneda embedding as a functor $よ : \Theta \to \mathcal{U}$. There is a subtlety here, due to the fact that $\mathcal{U}$ actually behaves as an ∞-category, so that $よ$ should be axiomatized as an ∞-functor, which is currently out of reach. However, it turns out to be sufficient in practice for our purposes to axiomatize it as a wild functor.

Postulate 2 (Yoneda embedding). *We suppose that we have a fully faithful wild functor* $\Theta \to \mathcal{U}$ *consisting of*

(A) a type $よP : \mathcal{U}$ *for every pasting scheme* P : PS,
(B) a function $よf : よP \to よQ$ *for every* $f : P \to_{\mathrm{PS}} Q$,
(C) an equality $よ\,\mathrm{id}_P = \mathrm{id}_{よP}$ *for every* P : PS,
(D) an equality $よg \circ よf = よ(g \circ f)$ *for every composable morphisms* $f : P \to_{\mathrm{PS}} Q$ *and* $g : Q \to_{\mathrm{PS}} R$,
(E) a proof that the map $\flat よ : \flat(P \to_{\mathrm{PS}} Q) \to \flat(よP \to よQ)$ *is an equivalence for* $P\,Q$ :: PS.

A type of the form $よP$ for some pasting scheme P is said to be *representable.*

Given a crisp type $A :: \mathcal{U}$ and a pasting scheme P : PS, the type A_P of P-*cells* of A is

$$A_P \quad \equiv \quad \flat(よP \to A)$$

Note that this type is only expected to be a space, as opposed to a general cellular space: this can be formulated here thanks to the $\flat$ modality. We simply write A_n for A_{O_n}. Note that (E) states that the space of P-cells of $よQ$ is precisely the space of maps $P \to_{\mathrm{PS}} Q$, i.e. $(よQ)_P \simeq (P \to_{\mathrm{PS}} Q)$. A consequence of the subsequent Postulate 6 is that the Yoneda functor preserves the terminal type:

Lemma 8. *We have* $よ[\,] = 1$.

Proof. Given a pasting scheme P, we have

$$\begin{aligned}
(よ[\,])_P &= \flat(よP \to よ[\,]) && \text{by definition of } (-)_P \\
&= \flat(P \to_{\mathrm{PS}} [\,]) && \text{by (E) of Postulate 2} \\
&= \flat 1 && \text{by property of } \Theta \\
&= \flat(よP \to [\,]) && \text{because } [\,] \text{ is terminal} \\
&= 1_P && \text{by definition of } (-)_P
\end{aligned}$$

By Postulate 6, we thus deduce that $よ[\,]$ is equivalent to 1. ◻

We thus have that $\flat A$ is the type of 0-cells of A as expected:

Corollary 9. *For any crisp type* A*, we have* $A_0 = \flat A$.

Taking cells in a crisp type A is functorial in the sense that a map $f : P \to_{\mathrm{PS}} Q$ induces a map $A_f : A_Q \to A_P$ (also sometimes noted $f^* : A_Q \to Q_P$), and a map $f :: A \to B$ induces a map $f_P : A_P \to A_Q$ (those are respectively given by pre- and post-composition). In particular, we have source and target maps $A_{\mathrm{src}} : A_P \to A_{\partial P}$ and $A_{\mathrm{tgt}} : A_P \to A_{\partial P}$, that we simply respectively write src and tgt in the following. We say that two P-cells are *parallel* when they have the same source and the same target (by convention, two 0-cells are always parallel).

3.7 Cellular cohesion

Now that we have introduced a type of pasting schemes along with the corresponding Yoneda embedding, we can formulate a new axiom which ensures that the $\flat$ modality actually behaves as explained in Section 3.3, by discarding all spaces in a presheaf, excepting the one corresponding to objects.

Given a type A and a pasting scheme P, we have a canonical map $A \to (よP \to A)$ sending an element $x : A$ to the constant function; this map can also be understood as the map $よt_P : A \simeq (よ[\,] \to A) \to (よP \to A)$ where $t_P : P \to [\,]$ is the terminal map in Θ. We say that A is *cellularly discrete* when this map is an equivalence for every P : PS. Such a type is thus local with respect to all representable types, in the sense of [53]. Recall from Section 3.3 that we have another notion of discreteness, which we call being $\flat$-*discrete* here: a type A is discrete in this sense when the canonical map $\flat A \to A$ is an equivalence. Following the cohesion axiom of [57, Axiom C0], we postulate that both notions of discreteness coincide:

Postulate 3 (Cellular cohesion). *For any type* $A :: \mathcal{U}$*,* A *is cellularly discrete if and only if it is* $\flat$*-discrete, and we simply say* discrete *for both.*

Proposition 10. *The types* PS *as well as* $P \to_{\mathrm{PS}} Q$ *for pasting schemes* P *and* Q *are discrete.*

Proof. We have seen in the proof of Proposition 7 that both types have decidable equality and are thus discrete by [57, Lemma 8.15]. In order to apply this lemma, we need to make sure that Axiom C1 of [57] is satisfied, which follows immediately from Postulate 3 and the fact that representable types are inhabited. The excluded middle is also listed as a requirement, but is not actually necessary for the direction of the implication we are using. ◻

As a consequence of the previous result, there is essentially no difference between PS and $\flat$ PS, thanks to which we will not need to be precise about whether the variables for pasting schemes are crisp or not. This also sheds light on the fact that the type PS does not correspond to the category Θ: although it is actually a category (by Proposition 10 and Theorem 19), its internal morphisms are trivial and thus not the morphisms of pasting schemes.

3.8 Suspension and hom types

We write $\mathcal{U}_{..} \equiv \Sigma(X : \mathcal{U}).(X \times X)$ for the type of *bipointed types*. Equivalently, a bipointed type is a type X equipped with two maps left, right $: 1 \to X$. Given a pasting scheme P, the type よP is canonically bipointed when equipped with the morphisms よ left, よ right : よ$[] \to$ よP, whose source can be considered to be 1 by Lemma 8. Given two bipointed types A and B, we write $A \to_{..} B$ for the type of *bipointed maps* between A and B, i.e. maps preserving the two distinguished elements.

We now want to axiomatize the existence of a suspension operation on types which can be thought of as a left adjoint $\mathrm{S} : \mathcal{U} \to \mathcal{U}_{..}$ to the hom-type functor from the ∞-category $\mathcal{U}_{..}$ to the ∞-category $\mathcal{U}$ which will be considered later. This operation should correspond to the one already defined for pasting schemes. Moreover, the characterization of hom-spaces toward suspensions in Proposition 4 suggests the following axiomatization:

Postulate 4 (Suspension). *We suppose that we have a crisp wild functor* $\mathrm{S} :: \mathcal{U} \to \mathcal{U}_{..}$ *and, for every* $P : \mathrm{PS}$ *with* $P \equiv [P_1, \dots, P_n]$*, an* interwining *map* $\beta_P ::$ よS$P \to_{..}$ SよP *which is an equivalence and such that the canonically induced map* $1 + \Sigma_i X_{P_i} + 1 \to (\mathrm{S}X)_P$ *is an equivalence. We moreover suppose that* S *preserves pushouts.*

Above, the canonical map $f : 1 + \Sigma_i X_{P_i} + 1 \to (\mathrm{S}X)_P$ sends the element of the left (resp. right) copy of 1 to the left (resp. right) canonical point of SX (which is bipointed), and the image of a cell $a : X_{P_i}$ in the middle summand is $\flat(\mathrm{S}(-) \circ \beta_{P_i} \circ$ よ$\pi_i)(a)$, i.e. the composite

$$よP \xrightarrow{よ\pi_i} よ\mathrm{S}P_i \xrightarrow{\beta_P} \mathrm{S}よP_i \xrightarrow{\mathrm{S}a_\flat} \mathrm{S}X_{P_i}$$

up to flattening. One of our main motivations for introducing suspension is to be able to define the hom of a type as a right adjoint:

Postulate 5 (Hom). *There is a crisp wild right adjoint to the suspension functor noted* hom $:: \mathcal{U}_{..} \to \mathcal{U}$*. The image of* (A, x, y) *is denoted* $\hom_A(x, y)$.

As expected, the 0-cells of $\hom_A(x, y)$ correspond to 1-cells in A from x to y.

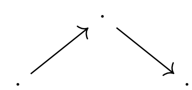
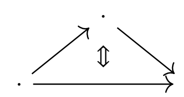
Lemma 11. *For* $A :: \mathcal{U}$ *and* $x, y :: A$*, we have*

$$\flat(\hom_A(x, y)) \quad = \quad \Sigma(a : A_1).(\mathrm{src}(a) = x) \times (\mathrm{tgt}(a) = y)$$

We sometimes write $x \to_A y$ instead of $\flat \hom_A(x, y)$. Further postulates are introduced in Section 4.2, after we define categories in order to motivate them.

4 Higher categories

We now identify, among types, those which should reasonably be considered as higher categories. Since types are interpreted as cellular-spaces, it is natural to introduce two conditions ensuring that they behave as higher categories, following the definition of Θ-spaces [50, 51], namely a Segal condition and a completeness condition. This approach is very similar in spirit to the one adopted in simplicial type theory [52] where "Rezk types" are defined as those being both Segal and complete. However, we need to formulate the Segal condition for pasting schemes instead of simplices, which can be done as follows, based on the definition in [39, Section 4.2.1.6]. Given a pasting scheme P, the type よP is expected to be a category (we will see that this is indeed the case in Theorem 30) and thus be closed under compositions. There is another type that we can associate to the type P, its realization, noted $\langle P \rangle$ and defined as the colimit of representables associated to the generators in P. The type $\langle P \rangle$ is similar to よP excepting that some compositions are not present. For instance, $\langle[2]\rangle$ corresponds to a simplicial 2-horn whereas よ$[2]$ corresponds to a 2-simplex:

$\langle[2]\rangle =$  よ$[2] =$ 

Given a pasting scheme P, we will have a canonical inclusion $\langle P \rangle \to$ よP and the Segal types will be defined as those having the right lifting property with respect to these maps.

4.1 Definition of higher categories

We begin by defining the realization of a pasting scheme as well as the canonical map to the corresponding representable type.

Definition 12. Given a pasting scheme $P \equiv [P_1, \dots, P_n]$, we write $\langle P \rangle$ for its *cellular realization*, defined as the colimit of the diagram

$$1 \xrightarrow{\text{right}} \mathrm{S}\langle P_1\rangle \xleftarrow{\text{left}} 1 \xrightarrow{\text{right}} \mathrm{S}\langle P_2\rangle \xleftarrow{\text{left}} \cdots \xrightarrow{\text{right}} \mathrm{S}\langle P_n\rangle \xleftarrow{\text{left}} 1 \tag{3}$$

Definition 13. Given a pasting scheme P, we define the *canonical map* $\mathrm{can}_P : \langle P \rangle \to$ よP from the realization of P to the corresponding representable type, by induction on P. Given pasting scheme $P \equiv [P_1, \dots, P_n]$ and an index i, we can define a map $\mathrm{S}\langle P_i \rangle \to$ よP as the composite

$$\mathrm{S}\langle P_i\rangle \xrightarrow{\mathrm{S}\,\mathrm{can}_{P_i}} \mathrm{S}よP_i \xrightarrow{\sim} よ\mathrm{S}P_i \xrightarrow{\subseteq_i} よP$$

The collection of these morphisms forms a cocone for the diagram (3) defining the cellular realization, thus inducing the desired map $\mathrm{can}_P : \langle P \rangle \to$ よP by universal property of the colimit.

In the case of the empty pasting scheme $[]$, we have that both $\langle[]\rangle$ and よ$[]$ are isomorphic to 1 (see Lemma 8), and we can consider that $\mathrm{can}_{[]}$ is identity on 1 without loss of generality. Similarly, for a pasting scheme of the form $[P]$, we can take $\mathrm{can}_{[P]}$ to be $\beta_P \circ \mathrm{S}\,\mathrm{can}_P$.

We can now define the property of being Segal as a right lifting property.

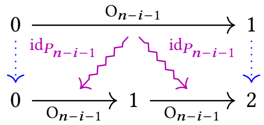
Definition 14. A type $A :: \mathcal{U}$ is *Segal* when for every pasting scheme $P : \mathrm{PS}$, the map

$$\flat(よP \to A) \to \flat(\langle P \rangle \to A)$$

induced by precomposition with can_P is an equivalence.

Suppose fixed a Segal type $A :: \mathcal{U}$. Given $P : \mathrm{PS}$, an element $a : A_P$ is called a *P-cell*, and its *source* and *target* are respectively the ∂P-cells $A_{\mathrm{src}}(a)$ and $A_{\mathrm{tgt}}(a)$, which we simply write as $\mathrm{src}(a)$ and $\mathrm{tgt}(a)$. Given $n \in \mathbb{N}$, an O_n-cell is simply called an *n-cell*.

We can define all the expected composition operations in a Segal type A. The terminal map $\tau : \mathrm{O}_1 \to_{\mathrm{PS}} \mathrm{O}_0$ induces, for any n, by suspension, a map $\mathrm{S}^n\tau : \mathrm{O}_{n+1} \to_{\mathrm{PS}} \mathrm{O}_n$ and thus a map $\mathrm{id}_n : A_n \to A_{n+1}$ sending an n-cell of A to its identity. Similarly, given $i < n$, consider the pasting scheme $P \equiv [\mathrm{O}_{n-i-1}, \mathrm{O}_{n-i-1}]$. Since suspension preserves pushouts, the realization $\langle \mathrm{S}^i P\rangle$ is the pushout of the diagram $\mathrm{O}_n \xleftarrow{\mathrm{tgt}} \mathrm{O}_i \xrightarrow{\mathrm{src}} \mathrm{O}_n$ and a map $\flat(\langle \mathrm{S}^i P\rangle \to A)$ thus corresponds to a pair of i-composable n-cells in A. By the Segal property, such a map extends to a map $\flat(よ\mathrm{S}^i P \to A)$ and thus induces an n-cell in A by precomposition with the i-th suspension of the map $\mathrm{O}_{n-i} \to P$ defined by $(j \mapsto 2j, (\mathrm{id}_{\mathrm{O}_{n-i-1}})_j)$, i.e.



We have thus defined a map $- *_i - : A_n \times_{A_i} A_n \to A_n$ which corresponds to the composition of n-cells in codimension i. We simply write $a * b$ for the composition of n-cells a and b in codimension $n-1$. Those operations have the expected source and target maps, e.g. for n-cells a and b, we have $\mathrm{src}(\mathrm{id}_n(a)) = \mathrm{tgt}(\mathrm{id}_n(a)) = a$, $\mathrm{src}(a * b) = \mathrm{src}(a)$, $\mathrm{tgt}(a * b) = \mathrm{tgt}(b)$, and so on. In particular, the composition operation induces a map

$$(x \to_A y) \to (y \to_A z) \to (x \to_A z)$$

still written $- * -$ for arbitrary parallel $(n-1)$-cells x, y, z.

Given a type $A :: \mathcal{U}$ and a cell $a : x \to_A y$, we say that a is *invertible* when the following type is inhabited:

$$\mathrm{isInv}(a) \equiv (\Sigma(b : y \to_A x).a * b = \mathrm{id}_x) \times (\Sigma(b : y \to_A x).b * a = \mathrm{id}_y)$$

We write $a \simeq_A b$ for the type of invertible maps in $a \to_A b$, and $A^{\simeq}_{n+1} \equiv \Sigma(a\ b : A_n).(a \simeq_A b)$ for the type of all invertible $(n+1)$-cells in A.

Definition 15. A type $A :: \mathcal{U}$ is *complete* when the canonical maps $(x =_{A_n} y) \to (x \simeq_{A_n} y)$ are equivalences for every dimension n and parallel n-cells $a, b :: A_n$.

The above property can be rephrased as the fact that the canonical map $A_n \to A^{\simeq}_{n+1}$ is an equivalence for every dimension n. Interestingly, the type of invertible maps can be represented as follows. Given $n \in \mathbb{N}$, we write E_{n+1} for the pushout

$$\begin{array}{ccc} よ\mathrm{S}^{n+1}[1] \sqcup よ\mathrm{S}^{n+1}[1] & \xrightarrow{[よ\mathrm{S}^n\alpha,\, よ\mathrm{S}^n\beta]} & よ\mathrm{S}^n[3] \\ {\scriptstyle よ\mathrm{S}^n t \sqcup よ\mathrm{S}^n t}\downarrow & & \downarrow \\ よ\mathrm{S}^n[1] \sqcup よ\mathrm{S}^n[1] & \longrightarrow & \ulcorner\ \mathrm{E}_{n+1} \end{array}$$

where $t : [1] \to [0]$ is the terminal map, the maps $\alpha, \beta : [1] \to [3]$ are given by $\alpha(0) \equiv 1$, $\alpha(1) \equiv 2$, $\beta(0) \equiv 1$ and $\beta(1) \equiv 3$, i.e.

and the above map $[よ\mathrm{S}^n\alpha, よ\mathrm{S}^n\beta]$ denotes the universal map induced by the coproduct from the two maps $よ\mathrm{S}^n\alpha$ and $よ\mathrm{S}^n\beta$. This object represents equivalences in the sense that we have $A^{\simeq}_{n+1} = \flat(\mathrm{E}_{n+1} \to A)$, see for instance [50, Proposition 10.1]. Since $(-) \to A$ sends pushouts to pullbacks, we have that $A^{\simeq}_{n+1}$ is the pullback $A^2_n \times_{A^2_{n+1}} A_{\mathrm{S}^n[3]}$. We thus have the following useful reformulation of completeness:

Lemma 16. *A type $A :: \mathcal{U}$ is complete when the following square is cartesian for every $n : \mathbb{N}$:*

$$\begin{array}{ccc} A^2_{n+1} & \xleftarrow{\langle A_{\alpha_n}, A_{\beta_n}\rangle} & A_{\mathrm{S}^n[3]} \\ {\scriptstyle \mathrm{id}_n \times \mathrm{id}_n}\uparrow & & \uparrow{\scriptstyle A_{\mathrm{S}^n t}} \\ A^2_n & \xleftarrow[\langle \mathrm{id}, \mathrm{id}\rangle]{} & \ulcorner\ A_n \end{array}$$

i.e. when the canonical map $\flat(よ\mathrm{O}_n \to A) \to \flat(E_{n+1} \to A)$ is an equivalence for all n.

Definition 17. A complete Segal type is called an (∞, ω)*-category*. We write $\mathrm{Cat}_{\infty,\omega}$ for their type.

4.2 Further postulates

Having defined categories, we show here general results about them. Namely, that all discrete types are categories (Section 4.3), that the hom of a category is still a category (Section 4.5), that representable types are categories (Section 4.6) and that categories are stable under finite limits and sums (Section 4.7). In order to be able to do so, we first need to introduce three more postulates, which enable one to effectively construct equivalences and manipulate types through representable types.

Since our types are to be interpreted as presheaves over Θ, we expect that they are entirely determined by their spaces of P-cells if we consider all pasting schemes P. We axiomatize this by the following postulate which will be very useful in order to build equivalences in practice, by defining them pointwise.

Postulate 6. *A map $f :: A \to B$ is an equivalence if and only if all the maps $f_P : A_P \to B_P$ are equivalences for $P : \mathrm{PS}$.*

In a category, an object A is *connected* when $\mathrm{Hom}(A, -)$ preserves finite coproducts [16]. For instance, the terminal set is the only connected object in the category of sets; in the category of topological spaces, connected spaces are those which are non-empty and connected in the usual sense. In the category of cellular spaces, all the representable objects are connected: the following postulate precisely asserts this. Another point of view on this is that colimits are computed pointwise in presheaf categories (such as cellular sets) [42, Corollary 5.1.2.3]: here, we only postulate that coproducts are pointwise.

Postulate 7. *For any $P : \mathrm{PS}$, the functor $(-)_P : \flat\mathcal{U} \to \flat\mathcal{U}$ preserves coproducts.*

Note that the previous discussion explains that it would have been reasonable to suppose more generally that $(-)_P$ preserves pushouts, or any family of small colimits. We have restricted ourselves to coproduct here only because this is what is necessary in our applications (see Section 4.7).

The following postulate roughly enforces a weak form of the density theorem, which states that every type is a colimit of representable types: here, we actually only require that a type is covered by representable ones. We recall that a map $f : A \to B$ is a *surjection* when for every $y : B$, we have that the fiber of f at y is merely inhabited, i.e. we have $\| \mathrm{fib}_f(y) \|_{-1}$, see [59, Definition 4.6.1].

Postulate 8. *For any type $X :: \mathcal{U}$, the canonical map*

$$\varepsilon : (\Sigma(P : \mathrm{PS}).\Sigma(x : X_P).よP) \to X$$

defined by $\varepsilon(P,x,p) \equiv x_\flat\, p$ *is a surjection.*

4.3 Discrete types are (∞,ω)-categories

We show here that discrete types are categories. This is intuitively expected because, by Postulate 3, a discrete type does not have non-trivial higher cells, and the conditions for being a category (Definition 17) are thus automatically satisfied.

Lemma 18. *If* $A :: \mathcal{U}$ *is a discrete type, then for all* $a,b :: A$*, we have* $\hom_A(a,b) \simeq (a =_A b)$.

Proof. By Postulate 6, it is enough to show that the canonical map $(a =_A b) \to \hom_A(a,b)$ is objectwise an equivalence. Given $P : \mathrm{PS}$, we have

$$\begin{aligned}\hom_A(a,b)_P &= \flat(\text{よ}SP \to_{..} (A,a,b)) && \text{by Postulate 5}\\ &= \Sigma(f : A_{SP}).(f_\flat(\text{left}) = a) \times (f_\flat(\text{right}) = b) && \text{by [57, Lemma 6.8]}\\ &= \Sigma(x : \flat A).(x_\flat = a) \times (x_\flat = b) && \text{because } A \text{ is discrete}\\ &= (a =_A b) && \text{by [57, Propositions 6.8 and 6.19]}\end{aligned}$$

and we conclude. ◻

Theorem 19. *Any discrete type* $A :: \mathcal{U}$ *is an* (∞,ω)*-category.*

4.4 Homotopy level is determined pointwise

The goal of this section is to characterize homotopy levels in our type theory. We namely show in Theorem 24 that a type A is an n-type if and only if so are all the types A_P.

We begin by observing that, because all types are covered by representable ones, in order to show that a property holds for all the elements of a type it is enough to show that it holds for all elements covered by a representable type.

Lemma 20. *For any* $X :: \mathcal{U}$ *and* $A : X \to \mathrm{Prop}$*, we have*

$$\big((P : \mathrm{PS}) \to (c : X_P) \to (s : \text{よ}P) \to A(c_\flat\, s)\big) \to (x : X) \to A(x)$$

Proof. Suppose given $x : X$. Since we are eliminating toward a proposition, by Postulate 8, we can assume that there is $P : \mathrm{PS}$, $c : X_P$ and $s : \text{よ}P$ such that $c_\flat\, s = x$. From the first argument, we deduce that $A(x)$ holds. ◻

We can now show the following universal property for 0-truncated crisp types: those are not only covered by representable types, but can actually be obtained as a canonical colimit of those (in the universe of 0-truncated types).

Proposition 21. *For any* $X :: \mathcal{U}$*, we can define maps* μ *and* ν *both from*

$$\Sigma(P,Q : \mathrm{PS}).\Sigma(\sigma : P \to_{\mathrm{PS}} Q).\Sigma(d : X_Q).\text{よ}P$$

and to

$$\Sigma(P : \mathrm{PS}).\Sigma(c : X_P).\text{よ}P$$

by $\mu(P,Q,\sigma,d,s) \equiv (P,\sigma^* d,s)$ *and* $\nu(P,Q,\sigma,d,s) \equiv (Q,d,(\text{よ}\sigma)\,s)$*. Then the canonical map* ε *of Postulate 8 is such that* $\varepsilon \circ \mu = \varepsilon \circ \nu$*, Moreover, for any* $Y :: \mathrm{Set}$*, precomposition by* ε *yield an equivalence*

$$\varphi : (X \to Y) \to \Sigma(f : (\Sigma(P : PS).\Sigma(c : A_P).\text{よ}P) \to Y).f\circ\mu = f\circ\nu$$

As a direct corollary, we deduce that equality of maps whose target is a set can be tested objectwise.

Corollary 22. *For any two maps* $f,g :: X \to Y$ *where* $X :: \mathcal{U}$ *and* $Y :: \mathrm{Set}$*,*

$$(f = g) \quad\leftrightarrow\quad ((P : PS) \to (c : X_P) \to f_P(c) = g_P(c))$$

Note that for second part of Proposition 21 to hold, and thus also for Corollary 22, it is crucial that Y is set-truncated. Otherwise, the specification of the map $f : X \to Y$ would require a specification of (non-propositional) homotopies for filling naturality squares.

A fundamental type is the one of spheres $\mathbb{S}^n$. By [59, eq. (6.5.2)], it can be defined as the coequalizer $\mathbb{S}^n \rightrightarrows 2 \longrightarrow \mathbb{S}^{n+1}$ where the two maps $\mathbb{S}^n \to 2$ are the two constant maps, and with the convention that $\mathbb{S}^{-1} \equiv 0$.

Proposition 23. *For every* n*, the* n*-sphere* $\mathbb{S}^n$ *is discrete.*

Proof. By [57, Theorem 6.21], discrete types contain 0, 2 and are closed under coequalizers. ◻

Finally, we can deduce the main theorem in this section, which states that truncation level for types can be tested pointwise.

Theorem 24. *Let* $X :: \mathcal{U}$*, then for any* $n \geq -2$*, we have*

$$\mathrm{isType}_n(X) \leftrightarrow \Pi(P : \mathrm{PS}).\,\mathrm{isType}_n(X_P)$$

Proof. First consider the left-to-right implication and suppose that X is an n-type. Given $P : \mathrm{PS}$, we have

$$\|X_P\|_n \equiv \|\flat(\text{よ}P \to X)\|_n = \flat\|\text{よ}P \to X\|_n$$

Since X is an n-type, then so is $\text{よ}P \to X$ by [59, Theorem 7.1.9], and thus $\|\text{よ}P \to X\|_n = \text{よ}P \to X$ by [59, Corollary 7.3.7]. We thus have $\|X_P\|_n = X_P$, from which we deduce that X_P is n-truncated by [59, Corollary 7.3.7] again.

Conversely, suppose that X_P is n-truncated for every $P : \mathrm{PS}$. By [59, Theorem 7.2.9], we have that X is an n-type if and only if the canonical map $(\mathbb{S}^{n+1} \to X) \to X$ (given by precomposition with the basepoint map $1 \to \mathbb{S}^{n+1}$) is an equivalence. By Postulate 6, this is equivalent to the canonical maps $(\mathbb{S}^{n+1} \to X)_P \to X_P$ being an equivalence for every $P : \mathrm{PS}$. Moreover, we have

$$\begin{aligned}(\mathbb{S}^{n+1} \to X)_P &\equiv \flat(\text{よ}P \to \mathbb{S}^{n+1} \to X)\\ &= \flat(\mathbb{S}^{n+1} \to \flat(\text{よ}P \to X))\\ &\equiv \flat(\mathbb{S}^{n+1} \to X_P)\end{aligned}$$

which uses (in addition to permuting arguments) the fact that we have $\flat(B \to \flat A) = \flat(B \to A)$ when B is discrete [57, Corollary 6.15], which is the case here for $\mathbb{S}^{n+1}$ by Proposition 23. We thus have that X is an n-type if and only if the canonical map $\flat(\mathbb{S}^{n+1} \to X_P) \to X_P$ is an equivalence. Since X_P is n-truncated, we have that the canonical map $(\mathbb{S}^{n+1} \to X_P) \to X_P$ is an equivalence and we conclude using the fact that $\flat$ preserves equivalences. ◻

4.5 Homs of categories

We show here that for any category, the hom-type between any two elements is again a category. We first establish that the canonical map between the realization of a pasting scheme and the corresponding representable is compatible with suspension. Recall that, given a pasting scheme P, the realization of SP is, by definition, the colimit reduced to SP and is thus canonically equivalent to it. Similarly, we have that $\text{よ}SP$ is equivalent to $S\text{よ}P$ by Postulate 4.

Lemma 25. *For any pasting scheme P : PS, there is a commutative square*

$$\begin{array}{ccc} \langle \mathrm{S}P\rangle & \xrightarrow{\mathrm{can}_{\mathrm{S}P}} & \text{よ}\mathrm{S}P \\ \wr\downarrow & & \downarrow\wr \\ \mathrm{S}\langle P\rangle & \xrightarrow[\mathrm{S}\,\mathrm{can}_P]{} & \mathrm{S}\,\text{よ}P \end{array}$$

Proof. Recall that the canonical map $\mathrm{can}_Q : \langle Q\rangle \to \text{よ}Q$ was introduced in Definition 13. In the particular case where $Q \equiv [P]$ is the suspension of a pasting scheme P, this map is

$$\mathrm{S}\langle P\rangle \xrightarrow{\mathrm{S}\,\mathrm{can}_P} \mathrm{S}\,\text{よ}P \xrightarrow{\sim} \text{よ}\mathrm{S}P \xrightarrow{\text{よ}\subseteq_P} \text{よ}Q$$

where $\text{よ}\subseteq_P$ is the identity, thus providing the commutative square, by definition of can_Q. ◻

Lemma 26. *If $A :: \mathcal{U}$ is a Segal type and $a, b :: A$, then for any pasting scheme $\mathrm{S}P$, the canonical map*

$$\flat(\text{よ}P \to_{..} (A, a, b)) \to \flat(\langle P\rangle \to_{..} (A, a, b))$$

is an equivalence

Proof. The proof mainly consists in commuting $\flat$ to Σ-types and identity types [57, Lemma 6.8 and Corollary 6.2], and using the bipointedness of can_P. ◻

Lemma 27. *Let $A :: \mathcal{U}$ be a Segal Type, and $a, b :: A$, then $\hom_A(a, b)$ is also Segal.*

Proof. We have

$$\begin{aligned} &\flat(\text{よ}P \to \hom_A(a, b)) \\ &\quad = \flat(\mathrm{S}\,\text{よ}P \to_{..} (A, a, b)) && \text{by the adjunction } \mathrm{S} \dashv \hom \\ &\quad = \flat(\text{よ}\mathrm{S}P \to_{..} (A, a, b)) && \text{by Postulate 4} \\ &\quad = \flat(\langle \mathrm{S}P\rangle \to_{..} (A, a, b)) && \text{by Lemma 26} \\ &\quad = \flat(\mathrm{S}\langle P\rangle \to_{..} (A, a, b)) && \text{by Lemma 25} \\ &\quad = \flat(\langle P\rangle \to \hom_A(a, b)) && \text{by the adjunction } \mathrm{S} \dashv \hom \end{aligned}$$

and we conclude. ◻

Theorem 28. *Let $A :: \mathcal{U}$ be a an (∞, ω)-category and $a, b :: A$. Then $\hom_A(a, b)$ is an (∞, ω)-category.*

Proof. By Lemma 27, we know that $\hom_A(a, b)$ is Segal, and it remains to prove its completeness. First observe that $\mathrm{E}_{n+1} \simeq \mathrm{S}\,\mathrm{E}_n$ by our assumption that S preserves pushouts Postulate 4.

$$\begin{aligned} &\flat(\text{よ}\,\mathrm{O}_n \to \hom_A(a, b)) \\ &\quad = \flat(\text{よ}\,\mathrm{O}_{n+1} \to \hom_A(a, b)) && \text{by the adjunction } \mathrm{S} \dashv \hom \\ &\quad = \flat(\mathrm{E}_{n+2} \to_{..} (A, a, b)) && \text{by completeness of } A \\ &\quad = \flat(\mathrm{E}_{n+1} \to_{..} \hom_A(a, b)) && \\ &\quad && \text{by } \mathrm{S} \dashv \hom \text{ and the previous observation} \end{aligned}$$

and we conclude. ◻

4.6 Representable types are (∞, ω)-categories

We show here that the representable types, i.e. those of the form $\text{よ}P$ for some pasting scheme P, are categories.

Proposition 29. *Given P : PS, the type $\text{よ}P$ is Segal.*

Theorem 30. *Given P : PS, the type $\text{よ}P$ is an (∞, ω)-category.*

Proof. The type $\text{よ}P$ is Segal by Proposition 29 and we turn to completeness. On the one hand, note that if $a, b : (\text{よ}P)_n$, then $a = b$ is propositional by Postulate 2 and Proposition 7. On the other hand, using Lemma 3, we have that if there is an equivalence $a \simeq_{\text{よ}P} b$, then $a = b$ and this equivalence is unique. We have therefore proved that $a = b$ and $a \simeq b$ are equivalent propositions, which gives the result. ◻

In the above proof, we have actually achieved a bit more than the theorem: we have shown that $\text{よ}P$ is skeletal. We have also proved that $1 \simeq \text{よ}[]$ is an (∞, ω)-category (which also follows from 1 being discrete, see Theorem 19).

4.7 Stability under sums and finite limits

We show here that categories are stable under finite limits and finite sums. Since we already know that the initial and terminal type are categories (by Theorem 19, for instance) all we have to show is stability under pullbacks and coproducts. We begin by characterizing the cells in those types before showing stability under pullbacks.

Lemma 31. *Consider categories $A, B, C :: \mathcal{U}$ be (∞, ω)-categories together with a pasting scheme P : PS. For any two maps $f :: B \to A$ and $g :: C \to A$, writing $B \times_A C$ for their pullback, we have*

$$(B \times_A C)_P \quad = \quad B_P \times_{A_P} C_P$$

Similarly, given two types $B, C :: \mathcal{U}$, we have

$$(B + C)_P \quad = \quad B_P + C_P$$

Lemma 32. *We have the following characterisation of invertible cells in pullbacks and sums:*

- *for any $A, B, C :: \mathcal{U}$ Segal-types, crisp maps $f :: B \to A$, $g :: C \to A$ and $n : \mathbb{N}$,*
$$(B \times_A C)^{\simeq}_{n+1} \quad \simeq \quad B^{\simeq}_{n+1} \times_{A^{\simeq}_{n+1}} C^{\simeq}_{n+1}$$
- *for any $B, C :: \mathcal{U}$ Segal-types and $n : \mathbb{N}$,*
$$(B + C)^{\simeq}_{n+1} \quad \simeq \quad B^{\simeq}_{n+1} + C^{\simeq}_{n+1}$$

Proposition 33. *Let $A, B, C :: \mathcal{U}$ be three (∞, ω)-categories with two crisp maps $f :: B \to A$ and $g :: C \to A$. Then their pullback $B \times_A C$ is again an (∞, ω)-category.*

Proof. We first show that $B \times_A C$ is Segal. Given P : PS, we have

$$\begin{aligned} \flat(\langle P\rangle \to B \times_A C) &\simeq \flat((\langle P\rangle \to B) \times_{\langle P\rangle \to B} (\langle P\rangle \to C)) \\ & && \text{by universal property} \\ &\simeq \flat(\langle P\rangle \to B) \times_{\flat(\langle P\rangle \to B)} \flat(\langle P\rangle \to C) \\ & && \text{by [57, Theorem 6.10]} \\ &\simeq B_P \times_{A_P} C_P && \text{by Segalness of } A, B \text{ and } C \\ &\simeq (B \times_A C)_P && \text{by Lemma 31} \end{aligned}$$

For completeness, given $n : \mathbb{N}$, we have:

$$\begin{aligned}(B \times_A C)_n &\simeq (B_n \times_{A_n} C_n)_n && \text{by Lemma 31}\\ &\simeq (B^{\simeq}_{n+1} \times_{A^{\simeq}_{n+1}} C^{\simeq}_{n+1})_n && \text{by completeness of } A, B \text{ and } C\\ &\simeq (B \times_A C)^{\simeq}_{n+1} && \text{by Lemma 32}\end{aligned}$$

and we conclude. □

In order to show stability under coproducts, we first show that realizations of pasting schemes are connected.

Lemma 34. *Given P : PS and $X, Y :: \mathcal{U}$, $\langle P\rangle$ is connected, in the sense that any crisp map $f :: \langle P\rangle \to X + Y$ factors through X or Y.*

Proof. We show the stronger property that $\mathrm{S}^n\langle P\rangle$ is connected for any $n : \mathbb{N}$ and $P \equiv [P_1, \ldots, P_k]$, by induction on P. Since suspension preserves pushouts by Postulate 4, we have

$$\mathrm{S}^n\langle P\rangle = \mathrm{S}^{n+1}\langle P_1\rangle \sqcup_{よ\,\mathrm{O}_n} \ldots \sqcup_{よ\,\mathrm{O}_n} \mathrm{S}^{n+1}\langle P_k\rangle$$

By Postulate 7 the $よ\,\mathrm{O}_n$ are connected, and by induction hypothesis the $\mathrm{S}^{n+1}\langle P_i\rangle$ are connected, and we conclude using the fact that connected objects are stable under pushout along an inhabited type. □

Proposition 35. *Given two (∞,ω)-categories $A, B :: \mathcal{U}$, their coproduct $A + B$ is again an (∞,ω)-category.*

We do not expect that (∞,ω)-categories are closed under pushouts of types, because it would requires to have composites for cells glued next to one another in the pushout: an example of a non-Segal pushout of categories is the horn $\langle [2]\rangle$, where the composite of the two 1-cells is missing.

5 Conclusion and perspectives

We have defined a type theory in which cellular spaces and Θ-spaces can be manipulated in a synthetic way, and have illustrated its usefulness by showing various properties of categories. It would be interesting to have an implementation in order to concretely experiment with this type theory, but there are various possible extensions and variations on this work that we would like to explore before embarking into implementation.

One of the main features of our type theory is the ability to speak about a higher category of presheaves over the category Θ. It seems that a large part of the work done here does not actually depend on Θ (developements depending on Postulates 2, 3 and 6 to 8) and would generalize to presheaves over any category satisfying suitable conditions. It would be interesting to investigate those conditions and explore the definition of type theories admitting semantics in arbitrary presheaf categories. In particular, this would enable one to approach synthetically other models of ∞-categories or higher algebraic structures (such as Γ-spaces).

Although our approach so far has allowed us to define basic constructions in higher category theory, it crucially lacks functoriality properties. In homotopy type theory, such properties follow for free from the univalence axiom which ensures that every construction is homotopy invariant. For instance, the type of identities $x =_A y$ between two elements of A is functorial in x and y, thanks to the transport along other paths of A: any pair of paths $p : x' =_A x$ and $q : y =_A y'$ induces, by transport, a map $x =_A y \to x' =_A y'$, which corresponds to the composition of paths. In our directed setting, we expect that we could similarly associate to every 1-cells $p : \hom_A(x', x)$ and $q : \hom_A(y, y')$ a map $\hom_A(x, y) \to \hom_A(x', y')$, and more generally that we can see $\hom_A$ as a map of type $A^{\mathrm{op}} \times A \to \mathcal{U}$. This does not seem to be possible with our current definition of $\hom_A$, because it requires the variables x and y to be crisp, thus restricting them to groupoids.

To solve this issue and related ones, it would be interesting to formally introduce a type $X \rightharpoonup Y$ corresponding to the category of functors and oplax natural transformations. Its usefulness can be explained as follows. Writing $\mathbb{I} \equiv よ[1]$ for the formal arrow, the type $\mathbb{I} \to A$ is not the correct space of 1-cells in A (it only has the right core) because it does not capture the higher cells. However, the type $\mathbb{I} \rightharpoonup A$ should be much better behaved, and should enable us to define correct notion of hom-type or slices. It would also be interesting to axiomatize the left adjoint to this arrow type, which should correspond to the Crans-Gray tensor product of categories [39, Section 4.3], noted ⊗. This tensor product can be thought of as a variant of the cartesian product where various squares are filled with directed arrows instead of isomorphisms, so that it is also better behaved with respect to higher cells. We expect that the suspension SA can be recovered as a suitable quotient of $\mathbb{I} \otimes A$, and that the notion of (co)cartesian fibrations could be defined using it, thus paving the way toward a formalization of the (∞,ω)-categorical Yoneda lemma.

In order to extend our type theory with new type formers such as the tensor product and its right adjoints, we need to account for the presence of two different tensor products (the Crans-Gray tensor product ⊗ and the cartesian tensor product ×), whose presence should be reflected in the structure of contexts, which shall now have two distinct kinds of "commas" corresponding to the two products. This suggests investigating a variant of our type theory based on bunched logic [48, 54, 55], whose aim is precisely to take this kind of situation into account. The most subtle part here consists in correctly handling the interaction between bunched logic and dependent types, which is made simpler in our case because dependencies in ⊗-types is not needed. Another peculiarity of our setting is that ⊗ is non-symmetric and semi-cartesian (thus allowing for weakening).

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A Omitted proofs

Proof of Lemma 3. We write $P \equiv [P_1, \cdots, P_m]$ and proceed by induction on the dimension n. If $n = 0$, the cell a induces inequality $(a \circ \mathrm{src}) \le (a \circ \mathrm{tgt})$ as elements of $[m]$. Similarly $(b \circ \mathrm{src}) \le (b \circ \mathrm{tgt})$ and we deduce the result. Now, consider the inductive case, with $n > 0$. We may write $a \equiv (f, (\sigma_i)_{i \in I})$ and $b \equiv (g, (\tau_j)_{j \in J})$. First, note that $f(0) = g(0)$ and $f(1) = g(1)$ because $\mathrm{src} : \mathrm{O}_n \to \mathrm{O}_{n+1}$ preserves endpoints (because $n > 0$). We deduce that $I = J$ and $f = g$. Then, for each $i \in I$, we have $\sigma_i \circ \mathrm{src} = (a \circ \mathrm{src})_i$ (because $n > 0$), and similarly, we have $\tau_i \circ \mathrm{tgt} = (b \circ \mathrm{tgt})_i$, so that $\sigma_i \circ \mathrm{src} = \tau_i \circ \mathrm{tgt}$. Exchanging the roles of σ_i and τ_i above also yields $\tau_i \circ \mathrm{src} = \sigma_i \circ \mathrm{tgt}$. Hence, by induction hypothesis, we have $\sigma_i \circ \mathrm{src} = \sigma_i \circ \mathrm{tgt}$ and $\sigma = \tau$. In particular, we already have $a = b$. Finally, using the definition of src and tgt again, we have $(a \circ \mathrm{src})_i = (a \circ \mathrm{tgt})_i$ for each i, whence $a \circ \mathrm{src} = a \circ \mathrm{tgt}$. □

Proof of Proposition 4. The suspension of a pasting scheme produces a pasting scheme which is canonically bipointed so that suspension extends as an operation $\mathrm{S} : \Theta \to \Theta_{..}$. Moreover, the Yoneda embedding $\text{よ} : \Theta \to \mathrm{Psh}\,\Theta$ canonically extends as a map $\text{よ} : \Theta_{..} \to (\mathrm{Psh}\,\Theta)_{..}$ (a pasting scheme is bipointed when equipped with two maps $l, r : [] \to P$, and those are sent to maps $\text{よ}l, \text{よ}r : 1 \cong \text{よ}[] \to \text{よ}P$ by the Yoneda functor, making its image bipointed). We thus have a composite functor

$$\Theta \xrightarrow{\mathrm{S}} \Theta_{..} \xrightarrow{\text{よ}} (\mathrm{Psh}\,\Theta)_{..}$$

Moreover, the target category $(\mathrm{Psh}\,\Theta)_{..}$ is canonically isomorphic to the category $[\Theta, \mathcal{S}_{..}]$ of presheaves enriched in bipointed spaces. By left Kan extension, it induces a functor

$$\mathrm{Lan}_{\text{よ}}(\text{よ} \circ \mathrm{S}) : \mathrm{Psh}\,\Theta \to (\mathrm{Psh}\,\Theta)_{..}$$

and, by the usual formulas in enriched categories [34], the image of a presheaf $X \in \mathrm{Psh}\,\Theta$ can be computed as the following coend in $(\mathrm{Psh}\,\Theta)_{..}$:

$$\mathrm{Lan}_{\text{よ}}(\text{よ} \circ \mathrm{S})(X) \quad = \quad \int^{Q \in \Theta} X_Q \cdot \text{よ}\mathrm{S}Q$$

Namely, the category $(\mathrm{Psh}\,\Theta)_{..} \cong [\Theta, \mathcal{S}_{..}]$ is copowered over $\mathcal{S}_{..}$, where the copower of $A : \mathcal{S}_{..}$ and $X \in (\mathrm{Psh}\,\Theta)_{..}$ is the presheaf $(A \cdot X)$ in $(\mathrm{Psh}\,\Theta)_{..}$ such that, for $P \in \Theta$, the space $(A \cdot X)_P$ obtained from $A \times X_P$ by quotienting both subspaces $A \times \{l\}$ and $A \times \{r\}$ to a point (in particular, when A is the interval, this is precisely the topological suspension of X_P). We write $-_{++} : \mathcal{S} \to \mathcal{S}_{..}$ for the left adjoint to the forgetful functor, associating to a space A the free bipointed space $1 + A + 1$. This functor preserves colimits (as a left adjoint). We can then compute:

$$
\begin{aligned}
(\mathrm{S}X)_P &= (\mathrm{Lan}_{\text{よ}}(\text{よ} \circ \mathrm{S}))(X)_P && \text{by definition} \\
&= \int^{Q \in \Theta} X_Q \cdot \text{よ}\mathrm{S}Q_P && (A) \\
&= \int^{Q \in \Theta} X_Q \cdot \Theta(P, \mathrm{S}Q) && \text{by the Yoneda lemma} \\
&= \int^{Q \in \Theta} X_Q \cdot \textstyle\sum_i \Theta(P_i, Q)_{++} && (B) \\
&= \textstyle\sum_i^{\mathcal{S}_{..}} \displaystyle\int^{Q \in \Theta} X_Q \cdot \Theta(P_i, Q)_{++} && \text{sums commute with coends} \\
&= \textstyle\sum_i^{\mathcal{S}_{..}} \left(\displaystyle\int^{Q \in \Theta} X_Q \cdot \Theta(P_i, Q) \right)_{++} && -_{++} \text{ preserves colimits} \\
&= \textstyle\sum_i^{\mathcal{S}_{..}} X_{P_i ++} && \text{by the codensity formula} \\
&= \left(\textstyle\sum_i^{\mathcal{S}} X_{P_i} \right)_{++} && -_{++} \text{ preserves coproducts as a left adjoint} \\
&= 1 + \textstyle\sum_i^{\mathcal{S}} X_{P_i} + 1 && \text{by definition of } -_{++}
\end{aligned}
$$

Above, (A) is computed in $\mathcal{S}_{..}$, which is justified by the isomorphism $(\mathrm{Psh}\,\Theta)_{..} \cong [\Theta, \mathcal{S}_{..}]$ and the fact that coends are computed pointwise. For (B), by combinatorics on Θ, we have an isomorphism of spaces

$$\Theta(P, \mathrm{S}Q) \cong 1 + \textstyle\sum_i \Theta(P_i, Q) + 1 = \sum_i \Theta(P_i, Q)_{++}$$

and we conclude. □

Proof of Corollary 9. By Lemma 8, we have

$$A_0 \equiv \flat(\text{よ}[] \to A) = \flat(1 \to A) = \flat A$$ □

Proof of Lemma 11. We have

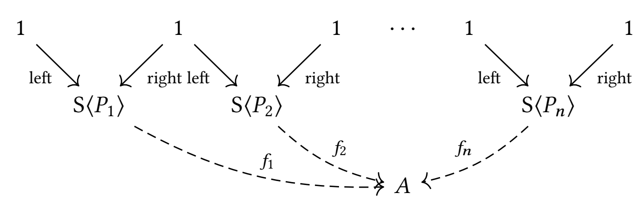
$$
\begin{aligned}
\flat(\hom_A(x, y)) &= \flat(1 \to \hom_A(x, y)) \\
&= \flat(\mathrm{S}1 \to_{..} (A, x, y)) && \text{by Postulate 5} \\
&= \flat(\Sigma(a : A_1).(\mathrm{src}(a) = x) \times (\mathrm{tgt}(a) = y)) \\
& && \text{by definition of bipointed maps} \\
&= \Sigma(a : A_1).(\mathrm{src}(a) = x) \times (\mathrm{tgt}(a) = y)
\end{aligned}
$$

The last line is due to the fact that A_1 is discrete and discrete types are closed under products [57, Lemma 8.4] and identity types [57, Lemma 8.5]. □

Proof of Theorem 19. Let us first show that A is Segal (Definition 14). Since A is (cellularly) discrete, this amounts to show that the constant map $\flat A \to \flat(\langle P \rangle \to A)$ is an equivalence. We reason by induction on the pasting scheme $P \equiv [P_1, \cdots, P_n]$. Note that for any index i, we have

$$
\begin{aligned}
&\flat(\mathrm{S}\langle P_i \rangle \to A) \\
&\quad = \flat\left(\Sigma(a, b : A).\langle P_i \rangle \to \hom_A(a, b)\right) && \text{by Postulate 5} \\
&\quad = \Sigma(a, b : \flat A).\mathrm{let}\ u^\flat, v^\flat = a, b \text{ in } \flat(\langle P_i \rangle \to \hom_A(u, v)) \\
&\quad && \text{because } \flat \text{ preserves } \Sigma\text{-types [57, Lemma 6.8]} \\
&\quad = \Sigma(a, b : \flat A).\mathrm{let}\ u^\flat, v^\flat = a, b \text{ in } \flat(\langle P_i \rangle \to u = v) \\
&\quad && \text{by Lemma 18} \\
&\quad = \flat\left(\Sigma(a, b : A).a = b\right) \\
&\quad && \text{by induction hypothesis and [57, Lemma 6.8]} \\
&\quad = \flat A && \text{by contractibility of singletons [59, Section 3.11]}
\end{aligned}
$$

Using the universal property of $\langle P \rangle$, a map $\flat(\langle P \rangle \to A)$ is equivalently a cocone



where, by the above, each f_i amounts to taking a point a_i in A, and the commutation of the squares to equalities $a_i = a_{i+1}$, from which we see that this data is contractible, and thus amounts to a point in A. This first part proves that any crisp discrete type A is a Segal one. Finally, we show that A is complete (Definition 15). By discreteness of A, for all $n \in \mathbb{N}$, we have $A_n \simeq A_0$, so that the square of Lemma 16 is a pullback. □

Proof of Proposition 21. For the first assertion, it follows directly from the fact that we have $(\sigma^* d)_\flat\, s = (d_\flat \circ \text{よ}\sigma)\, s$, for any P, Q, σ, d, s of expected type. We now prove the second assertion. First, we show that the fiber of φ is a proposition. Given a map

$$f : \big(\Sigma(P : \mathrm{PS}).\Sigma(c : A_P).\, \text{よ}P\big) \to Y$$

with two preimages $f^\dagger$ and $f^\ddagger : X \to Y$ under ϕ, we want to show that $f^\dagger = f^\ddagger$. By function extensionality, this amounts to show $\Pi(x : X).f^\dagger(x) = f^\ddagger(x)$. Since Y is a set, identity types are propositions, and by Lemma 20, we may show instead:

$$\Pi(R : \mathrm{PS}).\Pi(c : X_R).\Pi(s : \text{よ}R).f^\dagger(c_\flat\, s) = f^\ddagger(c_\flat\, s)$$

This holds because, by assumption, both handsides of the equality are identical to $f(R, c, s)$. We have shown that the fibers of φ are propositions, it remains to show that they are inhabited. In order for a factorization $f^\dagger : X \to Y$ of f to be well-defined, we only have to show that $f(P, c, s) = f(Q, d, t)$ whenever $c_\flat s = d_\flat t$. That is $\Pi(v : V).\Phi(v)$ where V is

$$\Sigma(P, Q : \mathrm{PS}).\Sigma(c : X_P).\Sigma(d : X_Q).\Sigma(s : \text{よ}P).\Sigma(t : \text{よ}Q).c_\flat\, s$$

and is thus equal to $d_\flat\, t$, and

$$\Phi(P, Q, c, d, s, t) \equiv (f(P, c, s) = f(Q, d, t))$$

We have that Φ is a proposition because Y is a set, so we may apply Lemma 20 once again. We thus have to prove $\Phi(m_\flat i)$ for some $m : \flat(\text{よ}R \to V)$ and $i : \text{よ}R$. Using the discreteness of PS, the idempotence of $\flat$ and the fully-faithfulness of よ, we reformulate the goal as the fact that we have $f(P, c, (\text{よ}s)(i)) = f(Q, d, (\text{よ}t)(i))$ whenever $s^* c = t^* d$ for every $R, P, Q : \mathrm{PS}$, $c : X_P$, $d : X_Q$, $s : R \to_{\mathrm{PS}} P$ and t: $R \to_{\mathrm{PS}} P$. But then, both handsides of the equality in the target are equal to $f(R, s^* c, i)$, whence the result. □

Proof of Proposition 29. By induction on $P : \mathrm{PS}$, we show the following more general property: for every natural number n and pasting scheme Q, we have $\flat(\mathrm{S}^n\langle P\rangle \to \text{よ}Q) \simeq \flat(\text{よ}\mathrm{S}^n P \to \text{よ}Q)$. Suppose given a pasting scheme $P \equiv [P_1, \cdots, P_m]$ such that the property holds for every P_i. We have

$$\begin{aligned}
&\flat(\mathrm{S}^n\langle P\rangle \to_{\mathrm{PS}} \text{よ}Q)\\
&\quad = \flat(\ldots \sqcup_{\mathrm{S}^n 1} \mathrm{S}^{n+1}\langle P_i\rangle \sqcup_{\mathrm{S}^n 1} \ldots \to \text{よ}Q) && \text{because S preserves pushouts}\\
&\quad = \ldots \times_{\flat(\mathrm{S}^n 1 \to \text{よ}Q)} \flat(\mathrm{S}^{n+1}\langle P_i\rangle \to \text{よ}Q) \times_{\flat(\mathrm{S}^n 1 \to \text{よ}Q)} \cdots && \text{by universal property of pushouts}\\
&\quad = \ldots \times_{\flat(\text{よ}\,\mathrm{O}_n \to \text{よ}Q)} \flat(\text{よ}\mathrm{S}^{n+1} P_i \to \text{よ}Q) \times_{\flat(\text{よ}\,\mathrm{O}_n \to \text{よ}Q)} \cdots && \text{by induction hypothesis}\\
&\quad = \ldots \times_{\flat(\mathrm{O}_n \to Q)} \flat(\mathrm{S}^{n+1} P_i \to Q) \times_{\flat(\mathrm{O}_n \to Q)} \cdots && \text{because よ is faithful}\\
&\quad = \flat(\text{よ}\mathrm{S}^n P \to_{\mathrm{PS}} \text{よ}Q) && \text{by Lemma 6 and よ full and faithful}
\end{aligned}$$

Specializing to the case $n = 0$ yields the desired isomorphism. □

Proof of Lemma 31. Let us show the property for pullbacks. We have

$$\begin{aligned}
&\flat(\text{よ}P \to B \times_A C)\\
&\quad = \flat((\text{よ}P \to B) \times_{\text{よ}P \to B} (\text{よ}P \to C)) && \text{by universal property}\\
&\quad = \flat(\text{よ}P \to B) \times_{\flat(\text{よ}P \to B)} \flat(\text{よ}P \to C) && \text{by [57, Theorem 6.10]}
\end{aligned}$$

The property for sums is precisely Postulate 7. □

Proof of Lemma 32. The property for pullbacks can be shown as follows:

$$\begin{aligned}
(B \times_A C)^{\simeq}_{n+1} &\simeq \flat(E_{n+1} \to B \times_A C) && \text{by Lemma 16}\\
&\simeq \flat((E_{n+1} \to B) \times_{E_{n+1} \to A} (E_{n+1} \to C)) && \text{by universal property}\\
&\simeq \flat(E_{n+1} \to B) \times_{\flat(E_{n+1} \to A)} \flat(E_{n+1} \to C) && \text{by [57, Theorem 6.10]}\\
&\simeq B^{\simeq}_{n+1} \times_{A^{\simeq}_{n+1}} C^{\simeq}_{n+1}
\end{aligned}$$

For sums, according to Lemma 31, an n-cell f of $A + B$ is either a n-cell of A or a n-cell of B. Then the data of a left (resp. right) inverse of f in $A + B$ will factor through the same component as f. Hence f will be invertible if and only if it is as a cell of A or as a cell of B. □

Proof of Proposition 35. Let us first show that $A + B$ is Segal. Given $P : \mathrm{PS}$, we have

$$\begin{aligned}
&\flat(\langle P\rangle \to A + B)\\
&\quad = \flat((\langle P\rangle \to A) + (\langle P\rangle \to B)) && \text{by Lemma 34}\\
&\quad = \flat(\langle P\rangle \to A) + \flat(\langle P\rangle \to B) && \text{by [57, Theorem 6.21]}\\
&\quad = A_P + B_P && \text{by Segalness of } A, B\\
&\quad = (A + B)_P && \text{by Lemma 31}
\end{aligned}$$

For completeness, given $n : \mathbb{N}$, we have

$$\begin{aligned}
(A + B)_n &= (A_n + B_n) && \text{by Lemma 31}\\
&= (A^{\simeq}_{n+1} + B^{\simeq}_{n+1}) && \text{by completeness of } A, B\\
&= (A + B)^{\simeq}_{n+1} && \text{by Lemma 32}
\end{aligned}$$

and we conclude. □

B Semantics of CellTT

In this appendix, we present the semantics of the type theory we have introduced, thus providing motivation for its axioms and their soundness.

B.1 Cellular spaces

We consider a model in *simplicial presheaves over* Θ, equipped with the Reedy model structure [28, Chapter 15], relative to the Quillen model structure on $\hat{\Delta}$, which coincides with the injective model structure [9]. We write $\mathrm{sPsh}(\Theta)$ for this model category. We write $\simeq$ for the equivalence relation generated weak equivalences (i.e. existence of a zig-zag of weak equivalences) and $\cong$ for isomorphism. We write $\text{よ} : \Theta \hookrightarrow \hat{\Theta} \hookrightarrow \mathrm{sPsh}(\Theta)$ for the Yoneda embedding and recall the most useful already known properties of this model structure, which may be found in [9, 28, 41, 50, 56].

Any object of $\mathrm{sPsh}(\Theta)$ is cofibrant, and cofibrations are the monomorphisms, see [50, Section 2.6] and [9, Proposition 3.15].

An object of sPsh(Θ) is *discrete* when it is set valued: any such object is fibrant [50, Section 2.6]. As in any Reedy category, a map $f : A \to B$ in sPsh(Θ) is a fibration if and only if, for each $P \in \mathrm{Ob}(\Theta)$, the induced map $A_P \to B_P \times_{M_P B} M_P A$ is a Kan fibration, where $M_P X = \lim_{(\Theta\downarrow P)^{\mathrm{op}}}(X|_{(\Theta\downarrow P)^{\mathrm{op}}})$ is the *matching object* of X [28, Definition 15.3.3]. Any two objects X, Y of sPsh(Θ) have a *mapping space* $\mathrm{Map}(X,Y) \in \hat{\Delta}$ and an *internal* Hom denoted $\underline{\mathrm{Hom}}(X,Y)$ in sPsh(Θ). Moreover, $\mathrm{Map}(よP \times X, Y) = \underline{\mathrm{Hom}}(X,Y)_P$ for any $P \in \Theta$, and $\mathrm{Map}(X,Y) = \underline{\mathrm{Hom}}(X,Y)_0$ are the global sections of $\underline{\mathrm{Hom}}(X,Y)$, see [50, Section 2.4] and [28, Section 11.7]. The presheaf $\mathrm{Map}(X,Y)$ is fibrant as soon as Y is because every object is cofibrant in sPsh(Θ) [28, Proposition 9.3.1]. The category sPsh(Θ) is a model of intentional type theory with dependent sums, dependent product, identity types, pushout types, truncations and a univalent universe for each innaccessible cardinal above $\aleph_0$ [56, Theorem 6.4]. We will call *set* any discrete simplicial set. Since limits and colimits are computed objectwise, sets are stable under limits and colimits.

The following lemma is useful to construct fibrations and thus interpret types. Namely, we recall that a type A should be interpreted as a fibrant object $[\![A]\!]$, and more generally $\Gamma \vdash A$ (a type A in a context Γ) as a fibration $[\![A]\!] \to [\![\Gamma]\!]$.

Lemma 36. *Any map $f : A \to B$ between discrete objects in* sPsh(Θ) *is a fibration.*

Proof. First, notice that if X is discrete, then for any $P \in \Theta$, we have that $M_P X$ is a set, as a limit of sets. Because of this, if A and B are discrete then so is $B_P \times_{M_P B} M_P A$. Therefore, the map $A_P \to B_P \times_{M_P B} M_P A$ is a Kan fibration as a map between discrete simplicial sets. □

B.2 Semantics of CellTT

B.2.1 Semantics of PS *(Postulate 1).* By Lemma 36, any objectwise discrete presheaf is a fibrant object. We can thus model the type PS of pasting schemes as the constant presheaf $[\![\mathrm{PS}]\!] \equiv P \mapsto \mathrm{Ob}(\Theta)$, which is objectwise discrete.

B.2.2 Semantics of $P \to_{\mathrm{PS}} Q$ (Postulate 1). We should model the type family $(P \to_{\mathrm{PS}} Q)_{P,Q}$ as a fibration over $[\![\mathrm{PS}]\!]^2$. By Lemma 36, it is enough to model $P \to_{\mathrm{PS}} Q$ for each $P, Q \in [\![\mathrm{PS}]\!]$, which can be done by $[\![P \to_{\mathrm{PS}} Q]\!] \equiv \mathrm{Hom}_\Theta(P,Q)$.

B.2.3 Semantics of よ : PS → $\mathcal{U}$ (Postulate 2). We now consider the Yoneda embedding introduced in Postulate 2. Using Lemma 36, it suffices to give an interpretation $[\![よP]\!]$ as a set valued presheaf for each P, which we define as $[\![よP]\!] \equiv よP$, the representable functor associated to P. The interpretation $[\![よ]\!]$ is now given by the fibration

$$\coprod_{P \in \mathrm{Ob}(\Theta)} よP \to \mathrm{Ob}(\Theta)$$

We should also give, for each $\sigma : P \to_{\mathrm{PS}} Q$, a map

$$[\![よ\sigma]\!] : よP \to よQ$$

which is given by $よ\sigma$ and the semantics of よ is therefore actually given by the Yoneda embedding. In particular, by functoriality of $よ : \Theta \to \mathrm{sPsh}(\Theta)$, it shows that the equality rules postulated in Postulate 2 hold on the nose: they could even be postulated as strict equalities.

For P and Q pasting schemes, $[\![P \to_{\mathrm{PS}} Q]\!]$ is a constant discrete presheaf with value $\mathrm{Hom}_\Theta(P,Q)$, and $\flat(よP \to よQ)$ is interpreted as the constant presheaf whose value is $\underline{\mathrm{Hom}}(よP, よQ)_0$. Both coincide since, by the Yoneda lemma, we have

$$\underline{\mathrm{Hom}}(よP, よQ)_0 = \mathrm{Map}(よP, よQ) \simeq \mathrm{Hom}_\Theta(P,Q)$$

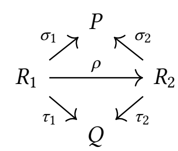
thus justifying the last point of Postulate 2.

B.2.4 Discreteness (Postulate 3). Recall that $\flat$ is interpreted as the endomorphism of sPsh(Θ) which to a presheaf X associates the constant presheaf $(P \mapsto X_0)$. A type interpreted as X in sPsh(Θ) is thus $\flat$-discrete whenever the map $\flat X \to X$ is a weak equivalence. That is, when for any $P \in \mathrm{Ob}(\Theta)$, the map $x \mapsto !^* x : X_0 \to X_P$ is a weak equivalence of simplicial sets, where $! : P \to []$ is the terminal arrow in Θ. On the other hand, the same type will be cellularly discrete whenever the maps

$$X_Q \to \mathrm{Map}(よP \times よQ, X)$$

are weak equivalences for each P and Q. By specializing this last condition to $Q = []$, we see that cellular discrete types are $\flat$-discrete. We now focus on the converse implication.

Definition 37. Let $P, Q \in \mathrm{Ob}(\Theta)$. We write $\mathcal{D}_{P,Q}$ (or $\mathcal{D}$ for short) for the category of elements of $よP \times よQ \in \hat{\Theta}$. The category $\mathcal{D}$ has as objects the spans $P \xleftarrow{\sigma} R \xrightarrow{\tau} Q$ and as morphisms the commutative diagrams



We write $F_{P,Q} : \mathcal{D} \to \mathrm{Psh}(\Theta) \hookrightarrow \mathrm{sPsh}(\Theta)$ for the functor sending such a morphism in $\mathcal{D}$ to $よ\rho : よR_1 \to よR_2$.

We refer to [28, Definition 15.10.1] for the notion of *category with fibrant constants*.

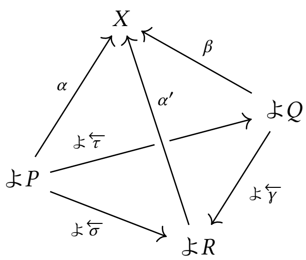
Lemma 38. *The category $\mathcal{D}$ is Reedy and has fibrant constants.*

Proof. The category $\mathcal{D}$ is Reedy as a category of elements of a presheaf over the Reedy category Θ [39, Proposition 1.1.2.6]. More precisely, the degree of $P \leftarrow R \to Q$ is defined as the degree of R, and the increasing (resp. decreasing) morphisms in $\mathcal{D}$ are those which are increasing (resp. decreasing) in Θ.

In order to see that $\mathcal{D}$ has fibrant constants, it is enough to show that for every object $\alpha \in \mathcal{D}$ the category $\partial(\alpha \downarrow \overleftarrow{\mathcal{D}})$ is either empty or connected [28, Proposition 15.10.2]. Let X be the presheaf $よP \times よQ$ (seen as a set valued presheaf), we will in fact show this result for X any such presheaf and $\mathcal{D}$ its category of elements. Suppose $\alpha : X_P$ is a P-cell of X. Then an object of $\partial(\alpha \downarrow \overleftarrow{\mathcal{D}})$ is the choice of a decomposition $\alpha = \overleftarrow{\sigma}^* \alpha'$ for some σ a strictly decreasing morphism of Θ and α' another cell of X.

- If α is non-degenerate, then there is no-such decomposition, hence $\partial(\alpha \downarrow \overleftarrow{\mathcal{D}})$ is empty.
- In the other case, there is a unique decomposition $\alpha = \overleftarrow{\sigma}^* \alpha'$ where α' is non-degenerate and σ strictly decreasing. Hence, for any other object $\alpha = \overleftarrow{\tau}^* \beta$ in $\partial(\alpha \downarrow \overleftarrow{\mathcal{D}})$, we may decompose uniquely β as $\overleftarrow{\gamma}^* \beta'$ where β' is non-degenerate and

$\overleftarrow{\gamma}$ is decreasing. By uniqueness of such decompositions, it follows that $\overleftarrow{\sigma} = \overleftarrow{\gamma} \circ \overleftarrow{\tau}$ and $\beta' = \alpha'$. That is, there is a (unique) commutative diagram



Hence, the decomposition $\alpha = \overleftarrow{\sigma}^* \alpha'$ is a terminal object in $\partial(\alpha \downarrow \overleftarrow{\mathcal{D}})$, so the matching category is connected. □

Lemma 39. *Let $R \in \Theta$ and consider the colimit*

$$\partial R \quad \equiv \quad \operatorname{colim}_{(Q \to R) \in \partial(\overrightarrow{\Theta} \downarrow R)} よQ$$

Then the canonical map $\partial R \to よR$ is a monomorphsim, identifying ∂R with the subpresheaf of morphisms $f : Q \to R$ which factors through a strictly increasing morphism.

Proof. Because colimits are computed pointwise and monomorphisms are also pointwise, we show that the map $(\partial R)_P \to (よR)_P$ is injective for each $P \in \mathrm{Ob}(\Theta)$. We use the explicit description of $(\partial R)_P$ as the quotient of $\coprod_{\overrightarrow{\sigma}: Q \to R} \mathrm{Hom}_\Theta(P, Q)$ under the relation $\sim$ such that $(\overrightarrow{\sigma} \circ \overrightarrow{\tau}, f) \sim (\overrightarrow{\sigma}, \overrightarrow{\tau} \circ f)$ for any $\tau \in \overrightarrow{\Theta}$: we have that the canonical map $(\partial R)_P \to (よR)_P$ sends the class $[(\overrightarrow{\sigma}, f)]$ to $\overrightarrow{\sigma} \circ f$. We then see that the image of $(\partial R)_P$ is the set of morphisms $P \to R$ which factor through a strictly increasing morphism $\overrightarrow{\sigma}$.

We now show the injectivity. Suppose there is a morphism $h : P \to R$ such that $h = \overrightarrow{\sigma} \circ f = \overrightarrow{\tau} \circ g$ for some strictly increasing morphisms $\overrightarrow{\sigma}$ and $\overrightarrow{\tau}$. Then factorizing f as $\overrightarrow{f} \circ \overleftarrow{f}$ and g as $\overrightarrow{g} \circ \overleftarrow{g}$, we get $\overrightarrow{\sigma} \circ \overrightarrow{f} = \overrightarrow{\tau} \circ \overrightarrow{g}$ and $\overleftarrow{f} = \overleftarrow{g}$ by uniqueness of the factorization of h. Hence, there is a chain of identifications

$$\begin{aligned}(\overrightarrow{\sigma}, f) &= (\overrightarrow{\sigma}, \overrightarrow{f} \circ \overleftarrow{f}) \sim (\overrightarrow{\sigma} \circ \overrightarrow{f}, \overleftarrow{f}) \\ &= (\overrightarrow{\tau} \circ \overrightarrow{g}, \overleftarrow{g}) \sim (\overrightarrow{\tau}, \overrightarrow{g} \circ \overleftarrow{g}) \\ &= (\overrightarrow{\tau}, g)\end{aligned}$$

□

Lemma 40. *The nerve $\mathrm{N}\,\mathcal{D}$ of the category $\mathcal{D}$ is contractible.*

Proof. Recall that the category $\mathcal{D}_{P,Q}$ is the comma $\Theta / よP \times よQ$. The category Θ is a strict test category [18, Example 5.12] and thus totally aspherical [18, Theorem 2.8], which implies precisely the contractibility of $\mathcal{D}_{P,Q}$. □

We can thus justify Postulate 3:

Theorem 41. *Both notions of discreteness coincide.*

Proof. We already have seen the first direction in Section B.2.4, and thus focus on the other one. Suppose that $\flat X \to X$ is a weak equivalence for some fibrant object X (we assume X to be fibrant because it models a type in the empty context). Then, for each $P \in \mathrm{Ob}(\Theta)$, we have a weak equivalence $X_0 \to X_P$. Recall that we want to show that the map $X_0 \to \mathrm{Map}(よP \times よQ, X)$ is a weak equivalence.

We have seen in Lemma 38 that $\mathcal{D}$ is Reedy with fibrant constants. Moreover, for any object $\alpha \equiv (P \leftarrow R \to Q)$, the latching category $\partial(\overrightarrow{\mathcal{D}} \downarrow \alpha)$ is isomorphic to $\partial(\overrightarrow{\Theta} \downarrow R)$, identifying the latching map $L_\alpha F_{P,Q} \to よR$ with the canonical map $\partial R \to よR$, which is a cofibration according to Lemma 39. Hence, by [28, Theorem 19.9.1], we have a weak equivalence

$$\operatorname{hocolim}_{\mathcal{D}} F_{P,Q} \xrightarrow{\sim} \operatorname{colim}_{\mathcal{D}} F_{P,Q} \cong よP \times よQ$$

Then, by [28, Theorem 19.4.4] (which applies because every object is cofibrant), we have a weak equivalence

$$\mathrm{Map}(よP \times よQ, X) \simeq \operatorname{holim}_{\mathcal{D}^{\mathrm{op}}} \mathrm{Map}(F_{P,Q}, X)$$

Moreover, there is for each $\alpha \equiv (P \leftarrow R \to Q)$ a weak equivalence

$$\mathrm{Map}(1, X) \cong X_0 \simeq X_R \cong \mathrm{Map}(F_{P,Q}(\alpha), X)$$

between fibrant objects (recall that X_0 and X_R are fibrant by [28, Proposition 18.5.3]). Hence, by [28, Theorem 18.5.3], there is a weak equivalence

$$\operatorname{holim}_{\mathcal{D}^{\mathrm{op}}} \mathrm{Map}(1, X) \simeq \operatorname{holim}_{\mathcal{D}^{\mathrm{op}}} \mathrm{Map}(F_{P,Q}, X)$$

By the previous reasoning, the righthand object is weakly equivalent to $\mathrm{Map}(よP \times よQ, X)$. Similarly, we have

$$\operatorname{holim}_{\mathcal{D}^{\mathrm{op}}} \mathrm{Map}(1, X) \simeq \mathrm{Map}(\operatorname{hocolim}_{\mathcal{D}} 1, X)$$

by [28, Theorem 19.4.4]. Finally, by definiton of the injective structure on $\mathrm{sPsh}(\Theta)$, and [28, Propositions 9.3.1 and 9.3.2], the pair of adjoint functors

$$\hat{\Delta} \underset{\mathrm{ev}_0}{\overset{\mathrm{const}}{\rightleftarrows}} \mathrm{sPsh}(\Theta) \qquad (\perp)$$

is a Quillen pair, so const preserves homotopy colimits [28, Theorem 19.4.5]. In particular, by [28, Proposition 18.1.6], we have

$$\operatorname{hocolim}_{\mathcal{D}} 1 \simeq \mathrm{const}(\mathrm{N}\,\mathcal{D}^{\mathrm{op}}) \simeq 1$$

because $\mathrm{N}\,\mathcal{D}$ in contractible by Lemma 40. Whence the weak equivalence

$$X_0 \to \mathrm{Map}(よP \times よQ, X)$$

□

B.2.5 Suspension functor in $\mathrm{sPsh}(\Theta)$ *(Postulate 4).* Let $\Theta_{..}$ be the category of bipointed pasting schemes and points-preserving maps, and similarly let $\mathrm{sPsh}(\Theta)_{..}$ denotes the simplicial category of bipointed simplicial presheaves over Θ (it is cocomplete, and even a model category [31, Proposition 1.1.8]). We may construct a suspension functor $\mathrm{S} : \mathrm{sPsh}(\Theta) \to \mathrm{sPsh}(\Theta)_{..}$ by Kan-extending the suspension operation $よ \circ \mathrm{S} : \Theta \to \Theta_{..} \hookrightarrow \mathrm{sPsh}(\Theta)_{..}$, as in [39, Section 4.2.1.11]. By construction, this extension restricts (up to a canonical isomorphism) to $よ \circ \mathrm{S}$ on the subcategory $\Theta \hookrightarrow \mathrm{sPsh}(\Theta)$, which yields the intertwining map.

B.2.6 Hom functor as a right adjoint of S *(Postulate 5).* We oberve that this Kan extension admits a right adjoint, which sends the bipointed presheaf (X, a, b) to $\mathrm{Hom}_X(a, b)$ defined by

$$\mathrm{Hom}_X(x_0, x_1)_P \equiv \mathrm{Map}_{..}(\mathrm{S}\,よP, (X, x_0, x_1))$$

where $\mathrm{Map}_{..}$ denotes the simplicial mapping space of bipointed maps, see also [39, Section 4.2.1.17].

B.2.7 Quillen adjunction and pushout preservation. We make the further observation that Hom sends a map of bipointed fibrant presheaves $f : (X, x_0, x_1) \to (Y, y_0, y_1)$ to the map

$$\mathrm{Hom}_X(x_0, x_1) \to \mathrm{Hom}_Y(y_0, y_1)$$

defined objectwise by postcomposition

$$f_* : \mathrm{Map}_{..}(\mathrm{S}\,よP, (X, x_0, x_1)) \to \mathrm{Map}_{..}(\mathrm{S}\,よP, (X, x_0, x_1))$$

This is a fibration (resp. a trivial fibration) when f is a fibration (resp. a trivial fibration) [28, Propositions 9.3.1 and 9.3.2]. The adjoint pair ($\mathrm{S} \dashv \mathrm{Hom}$) is thus a Quillen pair [28, Proposition 8.5.3]. Such a Quillen adjunction should yield the right notion of adjunction up to homotopy, and also ensures that $\mathrm{S} : \mathrm{sPsh}(\Theta) \to \mathrm{sPsh}(\Theta)_{..}$ preserves homotopy colimits, thus justifying the last assumption in Postulate 4.

B.2.8 Connectedness of representables (Postulate 7). If $[\![X]\!]$ is a (fibrant) object of $\mathrm{sPsh}(\Theta)$ interpreting a type $X :: \mathcal{U}$, then the interpretation of X_P for some $P : \mathrm{PS}$ is given by

$$[\![X_P]\!] = \mathrm{Map}(よP, [\![X]\!]) \cong [\![X]\!]_P$$

However, because sums are computed objectwise, the P-cells of a sum are canonically equivalent to the sums of the P-cells, see for instance [42, Section 5.1.2]. This justifies the connectedness of representable types.

B.2.9 Truncations are computed objectwise (Theorem 24). Let X be a fibrant object in $\mathrm{sPsh}(\Theta)$, by definition [42, Definition 5.5.6.1], X is n-truncated iff all the mapping spaces into it are n-truncated as Kan complexes. Note that this implies that each $X_P \simeq \mathrm{Map}(よP, X)$ is n-truncated. And conversely, if each X_P is n-truncated, then because every other object Y in $\mathrm{sPsh}(\Theta)$ is a colimit of representables, the mapping space $\mathrm{Map}(Y, X)$ is a limit of n-truncated Kan complexes, so that it is n-truncated [42, Proposition 5.5.6.5]. As a consequence, truncations may be computed objectwise.

B.2.10 Effective epimorphisms in $\mathrm{sPsh}(\Theta)$. An effective epimorphism in $\mathrm{sPsh}(\Theta)$ is the same as an effective epimorphism in its underlying 1-topos [42, Proposition 7.2.1.14], which is given by its homotopy category [42, Proposition 5.5.6.2], that is $\hat{\Theta}$, where the truncation is computed objectwise. Moreover, we know that effective epimorphisms coincide with epimorphisms in a 1-topos, and more precisely to objectwise surjections in the case of $\hat{\Theta}$.

B.2.11 Semantics of coverage (Postulate 8). Recall that PS is interpreted as a constant set valued presheaf. Writing X for the presheaf modeling the type X of Postulate 8, the sigma type

$$\Sigma(P : \mathrm{PS}).\Sigma(c : X_P).\,よP$$

will be interpreted as the coproduct

$$\coprod_{P \in \mathrm{Ob}(\Theta)} X_P \times よP$$

Hence the map

$$\coprod_{P \in \mathrm{Ob}(\Theta)} X_P \times よP \to X$$

will be an effective epimorphism if and only if all the maps

$$\coprod_{P \in \mathrm{Ob}(\Theta)} \pi_0(X_P) \times \mathrm{Hom}_\Theta(Q, P) \to \pi_0(X_Q)$$

are surjective for $Q \in \mathrm{Ob}(\Theta)$. Indeed, this is implied by surjectivity of the Q-th component

$$\pi_0(X_Q) \times \mathrm{Hom}_\Theta(Q, Q) \to \pi_0(X_Q).$$

This thus motivate our postulate of

$$\Sigma(P : \mathrm{PS}).\Sigma(c : X_P).\,よP \to X$$

being an effective epimorphism, assuming the type theoretic effective epimorphisms to indeed be modeled by effective epmorphisms in the higher categorical semantic.

B.2.12 Higher categories (Definition 17). Our definitions of Segalness (Definition 14) and completenness (Definition 15) mirrors the known definition of such concepts (see for instance [39, Section 4.2.1.6], where higher categories are defined as a localization of $\mathrm{sPsh}(\Theta)$). We thus expect that our type theoretic definition of (∞, ω)-categories coincide with the usual one in the above described model, although detailed verification is left for future work.

C Codiscrete types

In this section, we suppose that our type theory is equipped with a $\sharp$ modality, as axiomatized in [57, Section 3], and explore its properties in our setting. In terms of semantics, the functor $r : \mathrm{Psh}(\Theta) \to \mathcal{S}$ of Section 3.3 also admits a right adjoint, thus inducing a monad $\sharp$ on $\mathrm{Psh}(\Theta)$ which gives rise to an adjoint modality $\sharp \vdash \flat$ on types. The $\sharp$-modal types (the types A for which the canonical map $A \to \sharp A$ is an equivalence) are called *codiscrete*. We will see in Theorem 44 that their cells are entirely determined by their 0-skeleton. They thus have contractible hom-types and are to be thought as a directed counterpart of (-1)-truncated types. Among their properties, they form a reflexive subuniverse, and all of them are Segal types.

We recall from [57, Section 3] that a type A is codiscrete if and only if the canonical map $A \to \sharp A$ admits a retraction. We also recall the following useful fact, where the notion of reflexive subuniverse is defined in [59, Section 7.7].

Theorem 42. *The type of codiscrete types forms a reflexive subuniverse, when equipped with the $\sharp$ modality.*

In particular, codiscrete types are stable under identity-types, dependent sums and product. Finally, we recall the following fundamental property, which states that $\flat$ and $\sharp$ are internally adjoint to each other, up to flattening [57, Corollary 6.26]:

Theorem 43. *Given $A, B :: \mathcal{U}$, we have a natural equivalence*

$$\flat(\flat A \to B) \simeq \flat(A \to \sharp B)$$

The following theorem states that all the spaces A_P of cells in a type A are determined by their 0-skeleton A_0.

Theorem 44. *Let $A :: \mathcal{U}$ be a crisp type, then A is codiscrete if and only if the canonical map $\flat\,よP \to よP$ induces an equivalence $A_P \simeq \flat(\flat\,よP \to A)$ for every $P : \mathrm{PS}$.*

Proof. We have that A is codiscrete if and only if the canonical map $A \to \sharp A$ is an equivalence, if and only if the maps $A_P \to (\sharp A)_P$ are equivalences for $P : \mathrm{PS}$ (by Postulate 6). By Theorem 43, we have $(\sharp A)_P \equiv \flat(よP \to \sharp A) \simeq \flat(\flat\,よP \to A)$ and we conclude. □

Corollary 45. *Given a pasting scheme P, we have $A_P = A_0^m$ for some $m : \mathbb{N}$.*

Proof. Given a pasting scheme P, writing m for the number of 0-cells in $\text{よ}P$ (this is easily seen to be a finite set by Postulate 2 and the definition of Θ), we have $\flat\text{よ}P = m$, and therefore we have

$$\begin{aligned} A_P &\equiv \flat(\text{よ}P \to A) \\ &= \flat(\flat\text{よ}P \to A) && \text{by Theorem 44} \\ &= \flat(m \to A) && \text{by definition of } m \\ &= m \to \flat A && \text{since } \flat \text{ preserves products [57, Theorem 6.19]} \\ &= m \to A_0 && \text{by Corollary 9} \\ &= A_0^m \end{aligned}$$

and we conclude. □

Lemma 46. *Given a codiscrete crisp type $A :: \mathcal{U}$ and $a, b :: A$, the type $\hom_A(a, b)$ is contractible.*

Proof. Using Postulate 6, we may show the contractibility object-wise. Given $P : \mathrm{PS}$, we have

$$\begin{aligned} &\hom_A(a,b)_P \\ &\quad= \flat(\mathrm{S}\,\text{よ}P \to_{..} (A, a, b)) && \text{by Postulate 5} \\ &\quad= \Sigma(f : A_{\mathrm{S}P}).\mathrm{let}\ u^\flat = f \text{ in } \flat(u(\mathrm{left}) = a) \times \flat(u(\mathrm{right}) = b) \\ &&& \text{by [57, Lemma 6.8]} \\ &\quad= \Sigma(f : \flat(2 \to A)).\mathrm{let}\ u = f \text{ in } \flat(u(0) = a) \times \flat(u(1) = b) \\ &&& \text{by codiscreteness of } A \\ &\quad= \Sigma(x, y : \flat A).(x = a^\flat) \times (y = b^\flat) && \text{by [57, Theorem 6.1]} \\ &\quad= 1 && \text{by [59, Lemma 3.11.8]} \end{aligned}$$

and we conclude. □

Lemma 47. *Given a codiscrete crisp type $A :: \mathcal{U}$, we have $A_{n+1} \simeq A^{\simeq}_{n+1}$ for any $n : \mathbb{N}$.*

Proof. Using Lemma 16 and the definition of E_{n+1} we compute $A^{\simeq}_{n+1}$. Since A is codiscrete, using Theorem 43, one has

$$A^{\simeq}_{n+1} \simeq \flat(\flat\,\mathrm{E}_{n+1} \to A)$$

It is therefore enough to show that the map $\flat\,\text{よ}\,\mathrm{O}_{n+1} \to \flat\,\mathrm{E}_{n+1}$ is an equivalence, where the source $\flat\,\text{よ}\,\mathrm{O}_{n+1}$ is equivalent to 2. Recall that E_{n+1} is the colimit of the diagram

$$\begin{array}{ccccccc} & \text{よ}\,\mathrm{O}_{n+1} & & & \text{よ}\,\mathrm{O}_{n+1} & & \\ {\scriptstyle \mathrm{S}^n!}\swarrow & & \searrow{\scriptstyle \alpha_n} & {\scriptstyle \beta_n}\swarrow & & \searrow{\scriptstyle \mathrm{S}^n!} & \\ \text{よ}\,\mathrm{O}_n & & & \text{よ}\,\mathrm{S}^n[3] & & & \text{よ}\,\mathrm{O}_n \end{array}$$

and that, according to [57, Theorem 6.21], $\flat$ preserves pushouts. In the case $n = 0$, we have that $\flat\,\mathrm{E}_1$ is the colimit of the diagram

$$\begin{array}{ccccccc} & 2 & & & 2 & & \\ \swarrow & & \searrow & \swarrow & & \searrow & \\ 1 & & & 4 & & & 1 \end{array}$$

and this colimit is 2. In the case $n > 0$, we have that $\flat\,\mathrm{E}_{n+1}$ is the colimit of the diagram

$$\begin{array}{ccccccc} & 2 & & & 2 & & \\ {\scriptstyle \mathrm{id}}\swarrow & & \searrow{\scriptstyle \mathrm{id}} & {\scriptstyle \mathrm{id}}\swarrow & & \searrow{\scriptstyle \mathrm{id}} & \\ 2 & & & 2 & & & 2 \end{array}$$

which is, once again, 2. We deduce that the map $\flat\,\text{よ}\,\mathrm{O}_{n+1} \to \flat\,\mathrm{E}_{n+1}$ is an equivalence. □

Lemma 48. *Any crisp type $A :: \mathcal{U}$ is a Segal type.*

Proof. Since $\flat$ preserves pushouts [57, Theorem 6.21], we have that for any $P \equiv [P_1, \cdots, P_m]$ in PS, we have that $\flat\langle P\rangle$ is the colimit of the diagram

$$\begin{array}{ccccccccc} 1 & & 1 & & & \cdots & & & 1 \\ \searrow & \swarrow & \searrow & \swarrow & & & \searrow & \swarrow & \\ \flat\,\text{よ}\mathrm{S}P_1 & & & \flat\,\text{よ}\mathrm{S}P_2 & & & & \flat\,\text{よ}\mathrm{S}P_m & \end{array}$$

where for each index i, we have $\flat\,\text{よ}\mathrm{S}P_i = \flat\mathrm{S}\,\text{よ}P = 2$ by Postulate 4. Hence $\flat\langle P\rangle = \mathrm{Fin}_{m+1} = \flat\,\text{よ}P$. Then, by discreteness of A, we have $\flat(\langle P\rangle \to A) = \flat(\flat\langle P\rangle \to A) = \flat(\flat\,\text{よ}P \to A) = A_P$. □

Lemma 49. *A codiscrete crisp type $A :: \mathcal{U}$ is an (∞, ω)-category if and only if it is a proposition.*

Proof. Because A is codiscrete, by Corollary 45, for every $P : \mathrm{PS}$ we have $A_P = A_0^m$ where m is the number of 0-cells of P. Hence, by Theorem 24, A is a proposition iff A_0 is. Given $n > 0$, by Lemma 47 and Corollary 45, we have $A^{\simeq}_{n+1} = A_{n+1} = A_0^2 = A_n$. The only obstruction to completeness is thus for $n = 0$, i.e. A is complete iff $A^{\simeq}_1 = A_0$. By Lemma 47 and Corollary 45, we have $A^{\simeq}_1 = A_1 = A_0^2$, and A is thus complete iff the diagonal map $A_0 \to A_0^2 = A^{\simeq}_1$ is an equivalence. This occurs exactly when A_0 is propositional, that is iff A is. □

D (∞, n)-Categories

We have see in Section 4 that our type theory supports a notion of (∞, ω)-category. In particular, given $n : \mathbb{N}$, we expect that our type theory also supports a notion of (∞, n)-category, that is an (∞, ω)-category where every cell in dimension $m > n$ is (weakly) invertible. We briefly study this notion here.

Definition 50. Let $A :: \mathcal{U}$ be an (∞, ω)-category. Given $n : \mathbb{N}$, A is an *(∞, n)-category* when all its m-cells for $m > n$ are invertible, i.e. the canonical map $A_m \to A^{\simeq}_m$ is an equivalence. An $(\infty, 0)$-category is sometimes called an *∞-groupoid.*

Theorem 51. *A crisp type $A :: \mathcal{U}$ is discrete if and only if it is an ∞-groupoid.*

Proof. We know from Theorem 19 that any discrete type $A :: \mathcal{U}$ is an (∞, ω)-category. By completeness and discreteness we also have $A^{\simeq}_{n+1} \simeq A_n \simeq A_0 \simeq A_n$ for all n. Hence any discrete type is an ∞-groupoid.

Conversely, suppose $A :: \mathcal{U}$ is an ∞-groupoid. For any $n : \mathbb{N}$, by definition of ∞-groupoids and completeness, we have

$$A_{n+1} \simeq A^{\simeq}_{n+1} \simeq A_n$$

We may now prove that $A_P \simeq A_0$ by induction on P. Supposing given $P \equiv [P_1, \cdots, P_m]$ such that $A_{P_i} \simeq A_0$ for each i, we have

$$\begin{aligned} A_P &= \flat(\langle P\rangle \to A) && \text{by Segalness} \\ &= A_{\mathrm{S}P_1} \times_{A_0} \cdots \times_{A_0} A_{\mathrm{S}P_m} && \text{because } \flat \text{ preserves pullbacks} \\ &= \Sigma(a_0, \cdots, a_m : \flat A).\hom_A(a_0, a_1)_{P_1} \times \cdots \times \hom_A(a_{m-1}, a_m)_{P_m} \\ &&& \text{by } \mathrm{S} \dashv \hom \\ &= \Sigma(a_0, \cdots, a_m : \flat A).\hom_A(a_0, a_1)_0 \times \cdots \times \hom_A(a_{m-1}, a_m)_0 \\ &&& \text{by induction hypothesis} \end{aligned}$$

$$= \Sigma(a_0, \cdots, a_m : \flat A).\flat(a_0 = a_1) \times \cdots \times \flat(a_{m-1} = a_m) \qquad \text{because } A \text{ is an } \infty\text{-groupoid}$$
$$= A_0$$

The map $\flat A \to A$ is thus an equivalence by Postulate 6. ◻

Lemma 52. *An (∞, ω)-category $A :: \mathcal{U}$ is an $(\infty, n{+}1)$-category iff $\hom_A(a, b)$ is an (∞, n)-category for every $a, b :: A$.*

Proof. Let $A :: \mathcal{U}$ be a $(\infty, n{+}1)$-category and $a, b :: A$. Then for any $m > n$, we have

$$\begin{aligned} \hom_A(a,b)_m &= \flat(よ\, \mathrm{O}_{m+1} \to_{..} (A, a, b)) && \text{by S} \dashv \hom \\ &= \flat(\mathrm{E}_{m+1} \to_{..} (A, a, b)) && \text{by assumption} \\ &= \flat(\mathrm{S}\,\mathrm{E}_m \to_{..} (A, a, b)) && \text{because } \mathrm{S}\,\mathrm{E}_m = \mathrm{E}_{m+1} \\ &= \hom_A(a,b)^{\simeq}_m && \text{by S} \dashv \hom \end{aligned}$$

Conversely, suppose that $\hom_A(a, b)$ are (∞, n)-categories for all $a, b :: A$. Then for any $m > n$

$$\begin{aligned} A_{m+1} &= \flat(よ\, \mathrm{O}_{m+1} \to A) \\ &= \flat\left(\Sigma(a, b : A).\, よ\, \mathrm{O}_{m+1} \to_{..} (A, a, b)\right) \\ &= \Sigma(a, b : \flat A).\mathsf{let}\ u^\flat, v^\flat = a, b\ \mathsf{in}\ \flat(よ\, \mathrm{O}_{m+1} \to_{..} (A, u, v)) && \text{by } \flat \text{ commuting to } \Sigma \text{ and } \times \\ &= \Sigma(a, b : \flat A).\mathsf{let}\ u^\flat, v^\flat = a, b\ \mathsf{in}\ \hom_A(u, v)_m && \text{by S} \dashv \hom \\ &= \Sigma(a, b : \flat A).\mathsf{let}\ u^\flat, v^\flat = a, b\ \mathsf{in}\ \hom_A(u, v)^{\simeq}_m && \text{by hypothesis} \\ &= \Sigma(a, b : \flat A).\mathsf{let}\ u^\flat, v^\flat = a, b\ \mathsf{in}\ \flat(\mathrm{S}\,\mathrm{E}_m \to_{..} (A, u, v)) && \text{by S} \dashv \hom \\ &= \flat(\mathrm{E}_{m+1} \to A) && \text{because } \mathrm{S}\,\mathrm{E}_m = \mathrm{E}_{m+1} \\ &= A^{\simeq}_{m+1} \end{aligned}$$

and we conclude. ◻

Theorem 53. *(∞, n)-categories are closed under pullback and coproduct.*

Proof. This follows directly from Definition 50, Propositions 33 and 35 and Lemma 32. ◻