A type theory for cellular spaces

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Main Idea

- Simplicial Type Theory (*STT*)
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The interval

We add to the syntax a collection of shapes

$$\bullet \longrightarrow \bullet$$

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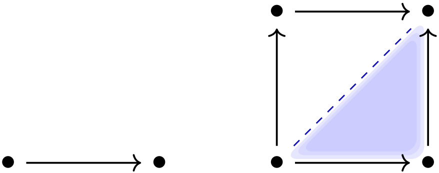

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$$\Delta^2 :\equiv \{\langle t_1, t_2\rangle : \mathbb{2}^2 \mid t_2 \leq t_1\}$$

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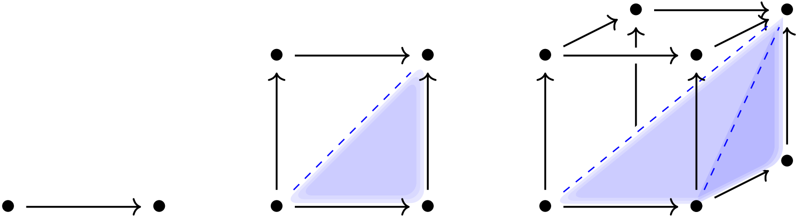

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S shape, and X type:

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So each type give rise to a *simplicial type !*

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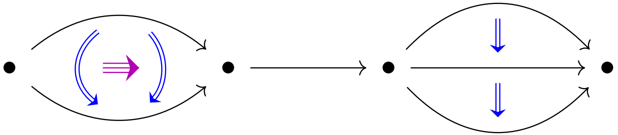
Reminiscent to Rezk-completes/univalent categories in HoTT.

The category Θ

- Objects: Pasting schemes.



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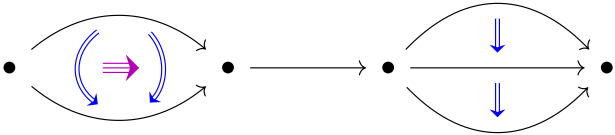

- Morphisms: Morphisms of strict ω-categories.

Some objects of Θ

$$\mathcal{O}_0 = *$$

Some objects of Θ

$$\bullet \qquad\qquad \bullet \longrightarrow \bullet$$

$$\mathcal{O}_0 = * \qquad\qquad \mathcal{O}_1 = [1]$$

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$$\begin{tikzcd} \bullet \arrow[r] & \bullet \end{tikzcd}$$

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$$\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \bullet \end{tikzcd}$$

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$$\begin{tikzcd} \bullet \arrow[r, bend left, ""{name=U, below}] \arrow[r, bend right, ""{name=D, above}] & \bullet \arrow[Rightarrow, from=U, to=D] \end{tikzcd}$$

$$\mathcal{O}_2$$

$$\begin{tikzcd} \bullet \arrow[r, bend left=60, ""{name=U, below}] \arrow[r, bend right=60, ""{name=D, above}] & \bullet \arrow[Rightarrow, from=U, to=D, bend right, ""{name=L}] \arrow[Rightarrow, from=U, to=D, bend left, ""{name=R, left}] \arrow[Rrightarrow, from=L, to=R] \end{tikzcd}$$

$$\mathcal{O}_3$$

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A combinatorial description of Θ

- Objects are lists of objects.

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$$* = [] \qquad\qquad \mathcal{O}_1 = [*] \qquad\qquad \mathcal{O}_2 = [\mathcal{O}_1] = [[*]]$$

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$$[n] = [*, *, \cdots, *]$$

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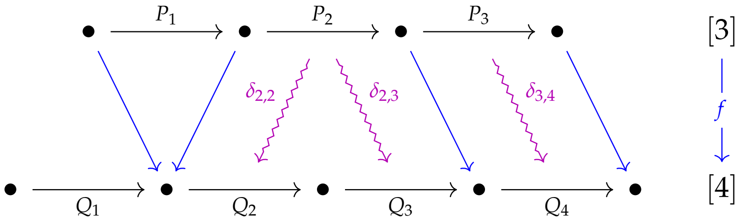

Hom Types in STT

In STT:

$$\mathsf{hom}_A(x,y) = \sum_{f : \mathsf{I} \to A} (f\,0 = x) \times (f\,1 = y)$$

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$$\begin{tikzcd} x \arrow[r, dash] & y \end{tikzcd}$$

Then:

$$\mathsf{hom}_{\mathsf{hom}_A(x,y)}(f, g) = \sum_{H : \mathsf{I}^2 \to A} \cdots$$

$$\begin{tikzcd} x \arrow[r, dash, "f"] \arrow[d, equal] & y \arrow[d, equal] \\ x \arrow[r, dash, "g"] & y \end{tikzcd}$$

Does not generalize to Θ

$$\mathsf{I} = よ([1])$$

Does not generalize to $\mathbf{\Theta}$

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2-cells of I^2 are pairs (x, y) of 2-cells of I.

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Hence I^2 is 1-categorical

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d] \arrow[dr] \arrow[dr, phantom, shift left=4ex, "="] \arrow[dr, phantom, shift right=4ex, "="] & \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet
\end{tikzcd}
\qquad
\begin{tikzcd}
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$$\mathsf{hom}_A(x, y) \neq \sum_{f : \mathsf{I} \to A} (f\,0 = x) \times (f\,1 = y)$$

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr] \arrow[dr, phantom, shift left=3, "="] \arrow[dr, phantom, shift right=3, "="] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr] \arrow[dr, phantom, shift left=3, "\swarrow"] \arrow[dr, phantom, shift right=3, "\swarrow"] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

Another approach

Workaround:

$$\begin{tikzcd} \mathbf{Psh}(\Theta) \arrow[rr, bend left, "\$"] \arrow[rr, phantom, "\perp"] & & \mathbf{Psh}(\Theta)_{\bullet\bullet} \arrow[ll, bend left, "\mathsf{hom}"] \end{tikzcd}$$

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Two subgoals:

- Defining a suspension \$.
- Postulating the adjunction.

($\$ \dashv \mathsf{hom}$) adjunction

$$(A \to \mathsf{hom}_B(x, y)) \overset{?}{\simeq} (\$A \to_{\bullet\bullet} (B, x, y))$$

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$$\underbrace{(A \to \mathsf{hom}_B(x, y))}_{\text{internal hom}} \neq \underbrace{(\$A \to_{\bullet\bullet} (B, x, y))}_{\text{internal hom}}$$

($\dashv$ hom) adjunction

$$\flat(A \to \mathsf{hom}_B(x, y)) = \flat(\$A \to_{\bullet\bullet} (B, x, y))$$

The $\flat$ modality

$$A :: \mathcal{U} \quad \rightsquigarrow \quad [\![A]\!] \in \mathbf{Psh}(\mathbf{\Theta})$$

$$\flat A : \mathcal{U} \quad \rightsquigarrow \quad [\![\flat A]\!] = [\![A]\!](*)$$

Semantic of $\flat$

$$\begin{tikzcd} \mathcal{S} \arrow[rr, bend left, "\delta"] \arrow[rr, phantom, "\perp"] & & \mathbf{Psh}(\Theta) \arrow[ll, bend left, "F \mapsto F(*)"] \end{tikzcd}$$

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$$
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\mathcal{S} \arrow[rr, bend left, "\delta"] \arrow[rr, phantom, "\perp"] & & \mathbf{Psh}(\Theta) \arrow[ll, bend left, "F \mapsto F(*)"] \arrow[loop right, "\flat"]
\end{tikzcd}
$$

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$$\frac{\Gamma | \cdot \vdash X : \mathcal{U}}{\Gamma | \cdot \vdash \flat X : \mathcal{U}}$$

CellTT

$$\text{CellTT} = \underbrace{\text{HoTT} \;+\; \text{Idempotent comodality}}_{\text{Crisp Type Theory}} + \text{Axioms}$$

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$$\sigma : P \to_{\mathsf{PS}} Q \quad \rightsquigarrow \quad \sigma^* : X_Q \to X_P$$

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Corollary 5.1.6.11. *Let $\mathcal{C}$ be an ∞-category which admits small colimits. Let S be a small simplicial set and $F : \mathcal{P}(S) \to \mathcal{C}$ a colimit preserving functor. Then F is an equivalence if and only if the following conditions are satisfied:*

(1) *The composition $f = F \circ j : S \to \mathcal{C}$ is fully faithful.*

(2) *For every vertex $s \in S$, the object $f(s) \in \mathcal{C}$ is completely compact.*

(3) *The set of objects $\{f(s) : s \in S_0\}$ generates $\mathcal{C}$ under colimits.*

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- $\left(\sum_{P:\mathsf{PS}} \sum_{c:X_P} よP\right) \to X$ is an effective epi ((-1)-truncated)
- $\flat$-discreteness is *cellular discreteness*:

$$\text{is-equiv}(\flat X \to X) \leftrightarrow \prod_{P:\mathsf{PS}} \text{is-equiv}(X \to (よ(P) \to X))$$

Levelwiseness of level

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- Case $n \geq -1$: LR implication: $\flat\|X\|_n \simeq \|\flat X\|_n$

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  $(X^{\mathbb{S}^{n+1}})_P \simeq \flat(よP \times \mathbb{S}^{n+1} \to X) \simeq \flat(\mathbb{S}^{n+1} \to X_P)$

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  $(X^{\mathbb{S}^{n+1}})_P \simeq \flat(\text{よ}P \times \mathbb{S}^{n+1} \to X) \simeq \flat(\mathbb{S}^{n+1} \to X_P)$
  $\flat(\mathbb{S}^{n+1} \to X_P) \simeq \flat X_P \simeq X_P$.

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We extend $ to $\mathcal{U}$.

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- $\text{よ}(\$P) = \$(\text{よ}P)$
- If $P = [P_1, P_2, \cdots, P_n]$:

$$(\$X)_P \cong \mathbb{1} + \left(\sum_{i\,:\,\mathsf{Fin}_n} X_{P_i} \right) + \mathbb{1}$$

$$\begin{tikzcd}
\bullet \arrow[r, dotted, "P_1, \cdots, P_{i-1}"] \arrow[dr] & \bullet \arrow[r, "P_i"] \arrow[d] \arrow[dr, phantom, "\Downarrow"] & \bullet \arrow[r, dotted, "P_{i+1}, \cdots, P_n"] \arrow[d] & \bullet \arrow[dl] \\
 & \bullet \arrow[r, "X"'] & \bullet &
\end{tikzcd}$$

Hom Types

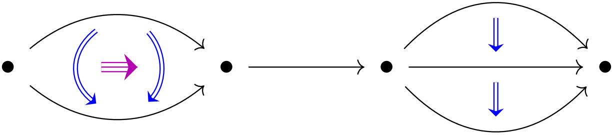
We postulate an adjunction

$$\flat(A \to \mathsf{hom}_B(x, y)) \cong \flat(\$A \to_{\bullet\bullet} (B, x, y))$$

Cellular Realization

$$\langle [P_1, \cdots, P_m] \rangle :\equiv$$

$$\operatorname{colim} \begin{tikzcd} & \mathbb{1} \arrow[dl] \arrow[dr] & & \mathbb{1} \arrow[dl] \arrow[dr] & & \mathbb{1} \arrow[dl] \arrow[dr] & \\ \$\langle P_1 \rangle & & \$\langle P_2 \rangle & & \cdots & & \$\langle P_m \rangle \end{tikzcd}$$



Segal Types

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Complete Types

We define a type E as the pushout

$$\begin{tikzcd}
よ[1] + よ[1] \arrow[r, "\alpha, \beta"] \arrow[d] & よ[3] \arrow[d] \\
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\end{tikzcd}$$

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We define is-complete:

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(∞, ω)-categories

As in Riehl-Shulman STT, there is a completeness condition.
$\mathsf{is\text{-}complete}(X)$

$$(\infty, \omega)\text{-categories} = \text{complete Segal Types.}$$

$$\mathsf{Cat}_{\infty,\omega} = \sum_{X:\mathcal{U}} \mathsf{is\text{-}Segal}(X) \times \mathsf{is\text{-}complete}(X)$$

Some elementary results

- (∞, ω)-categories are stable by pullback and sum.

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What's next ?

Main goal: Proving a Yoneda Lemma.

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- Crans-Gray tensor product $\boxtimes$.

Towards a bunched-affine variant of HoTT

$$\Gamma, \Delta \quad ::= \quad \diamond \mid \Gamma * \Delta \mid \Gamma, x : A$$

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$$\mathsf{hom}_A(x, y) = \sum_{f : よ[1] \multimap A} (\langle 0 \mid f = x) \times (\langle 1 \mid f = y)$$

Some rules

$$(\otimes\text{-}\textsc{Intro})\ \frac{\underline{\Lambda} \mid \Gamma \vdash a : A \qquad \underline{\Lambda} \mid \Delta \vdash b : B}{\underline{\Lambda} \mid \Gamma \otimes \Delta \vdash a \otimes b : A \otimes B}$$

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Some rules (2)

$$(\multimap\text{-INTRO})\ \frac{\underline{\Lambda} \mid \diamond \vdash A : \mathcal{U} \qquad \underline{\Lambda} \mid \diamond \vdash B : \mathcal{U} \qquad \underline{\Lambda} \mid \Gamma \otimes (x : A) \vdash b : B}{\underline{\Lambda} \mid \Gamma \vdash \varrho x.b : A \multimap B}$$

Some rules (2)

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The end

Thank you !

Questions ?