A Bunched Approach to Directed HoTT

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Samuel Mimram

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Outline

1. Homotopy type theory
2. Higher categories
3. Simplicial type theory
4. The problem of hom types
5. The bunched solution
6. From theory to application

Presentation

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- Constructive flavor ($\neq$ Classical Set Theory)
- Types behaves as spaces.
- Synthetic tool for higher algebra, homotopy theory...
- Suitable for formalizing mathematics / homotopy theory
- **Goal:** replace spaces by *directed spaces*

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- MARTIN-LÖF, 1972.
- conversely to Set Theory, no need for first order logic.
- “basic types” and “constructors” as building blocks.

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- **natural numbers** type, $\mathbb{N}$.
  $0 : \mathbb{N}$, $\mathsf{succ} : \mathbb{N} \to \mathbb{N}$.
  $n : \mathbb{N} \rightsquigarrow \mathsf{succ}(n) : \mathbb{N}$.
  eg., $\mathsf{succ}(\mathsf{succ}(\mathsf{succ}(0))) : \mathbb{N}$.

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- Sum type: $A + B$

Hierarchy of Universes

- Tarski (or Grothendieck) Universes. $\mathcal{U}$ is the "type of types"
- $\mathcal{U}$ closed under $\times$, $+$ and $\to$.
- Having $\mathcal{U} : \mathcal{U}$ leads to paradoxes

$$\mathcal{U}_0 : \mathcal{U}_1 : \mathcal{U}_2 : \ldots$$

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- $\mathbb{2}$:
  for any A, and $a_0 : A$, $a_1 : A$, $\mathsf{rec}_{\mathbb{2},A,a_0,a_1} : \mathbb{2} \to A$
  $\mathsf{rec}_{\mathbb{1},A,a_0,a_1}(0) \equiv a_0$ and $\mathsf{rec}_{\mathbb{1},A,a_0,a_1}(1) \cong a_1$.

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- $A + B$:
  for any types A, B, $f : A \to C$ and $g : B \to C$, $\mathsf{rec}_{A+B,C,f,g} : A + B \to C$
  $\mathsf{rec}_{A+B,C,f,g}(\mathsf{inl}(a)) \equiv f(a)$ and $\mathsf{rec}_{A+B,C,f,g}(\mathsf{inr}(b)) \equiv g(b)$.

More type formers

- Π-types:
  $A : \mathcal{U}, B : A \to \mathcal{U} \quad \rightsquigarrow \quad \prod_{a:A} B(a) : \mathcal{U}.$
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- Identity-types:
  $A : \mathcal{U}, x, y : A \quad \rightsquigarrow \quad x =_A y : \mathcal{U}.$
  $x : A \quad \rightsquigarrow \quad \mathsf{refl}_x : x =_A x.$

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Correspondance between types and logical propositions.

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$$\text{Logic} \hookrightarrow \text{Algebra}$$

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- Think of $x = y$ as the type of paths from x to y.

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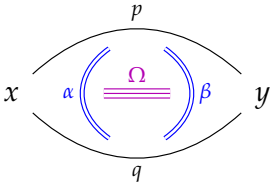
Theorem

Every type is equipped with a canonical structure of ∞-groupoid.

∞-groupoids

Spaces up to homotopy



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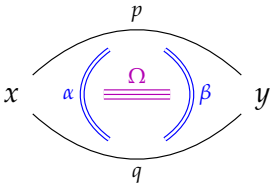

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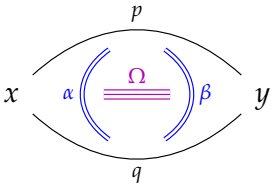

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Formalisation of homotopy theory
(Feudenthal, long exact sequences, sphere homotopy groups...).

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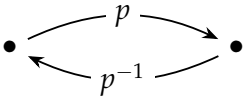
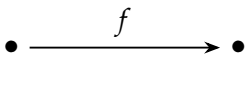


$x = y$



$\mathsf{hom}_A(x, y)$

∞-categories

"Directed" spaces up to homotopy

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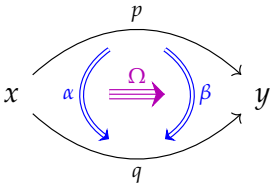

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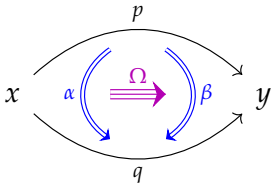

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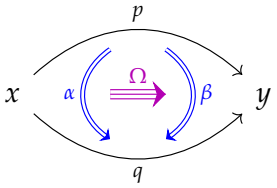

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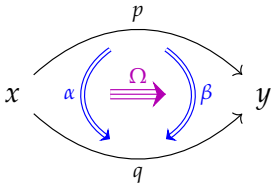

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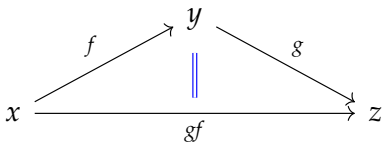
Rewriting, concurrency,
synthetic manipulation of higher categories.

Homotopy coherence

ordinary categories



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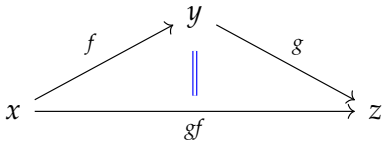

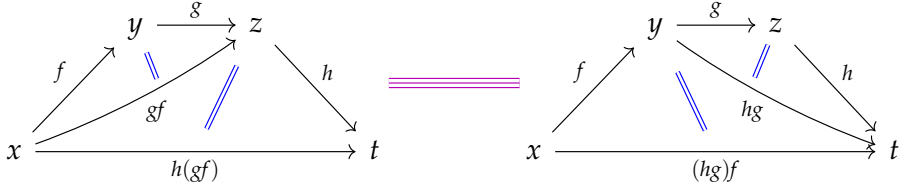

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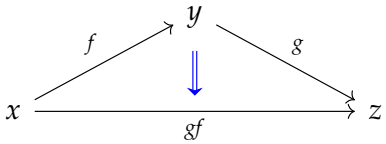

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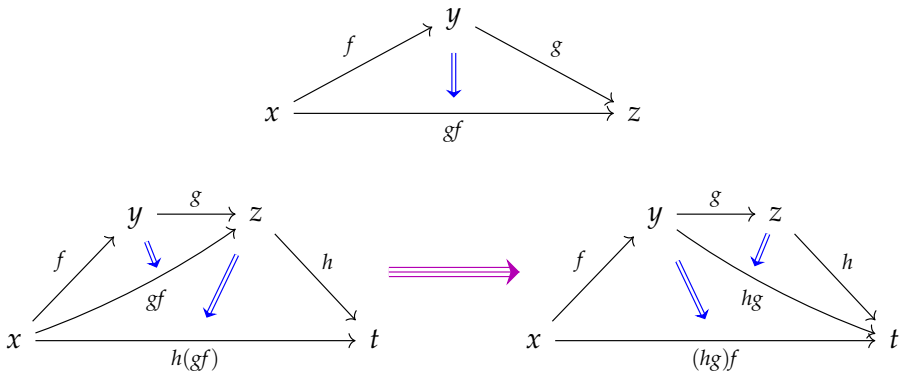
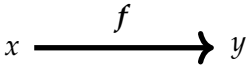


The nerve construction

Idea: categories $\rightsquigarrow$ simplicial set



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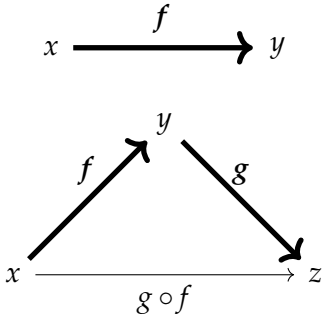

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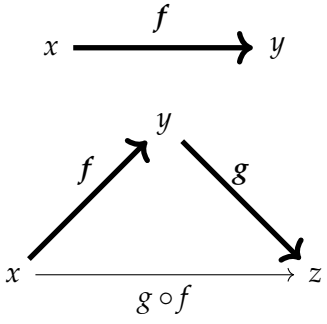

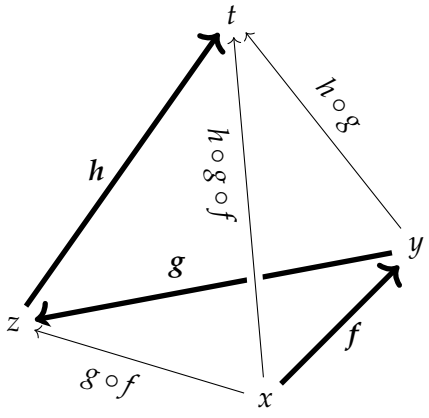

Simplicial spaces

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$\bullet \to \bullet$

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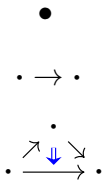

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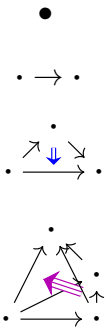

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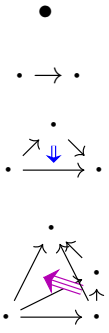
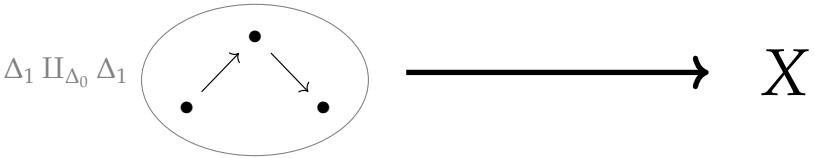
Space valued presheaf $X : \Delta^{\mathsf{op}} \longrightarrow \mathcal{S}$.



Segal spaces

Idea: encode composability as lifting properties.



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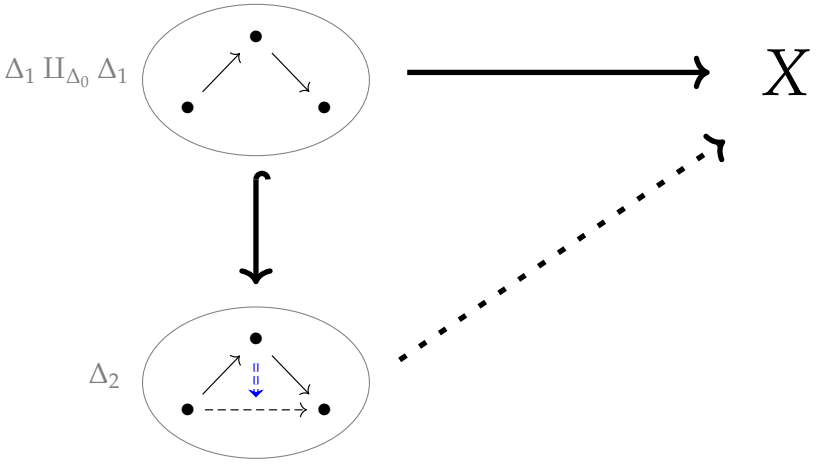

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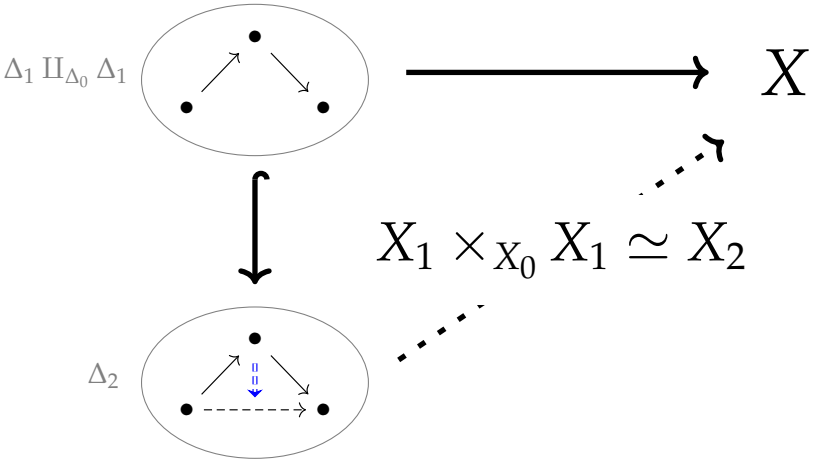

Segal spaces (2)

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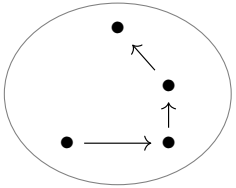
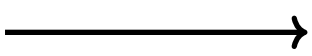
$\Delta_1 \amalg_{\Delta_0} \Delta_1 \amalg_{\Delta_0} \Delta_1$





X

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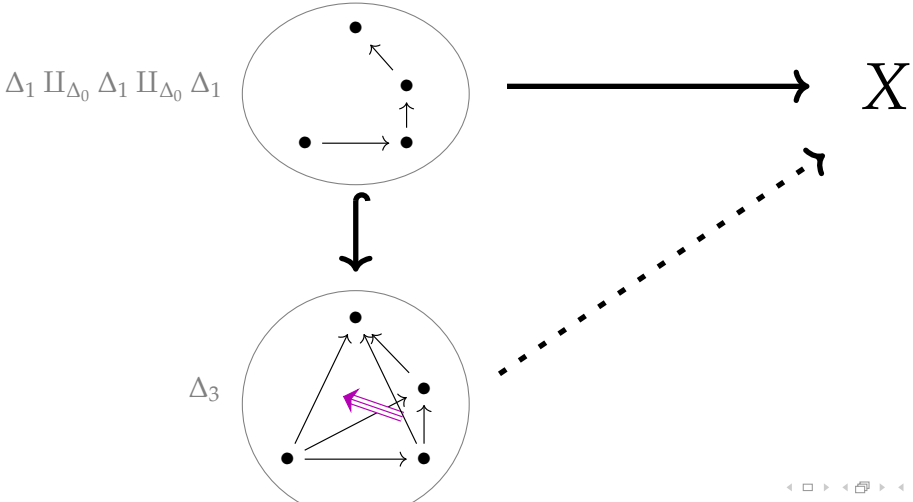

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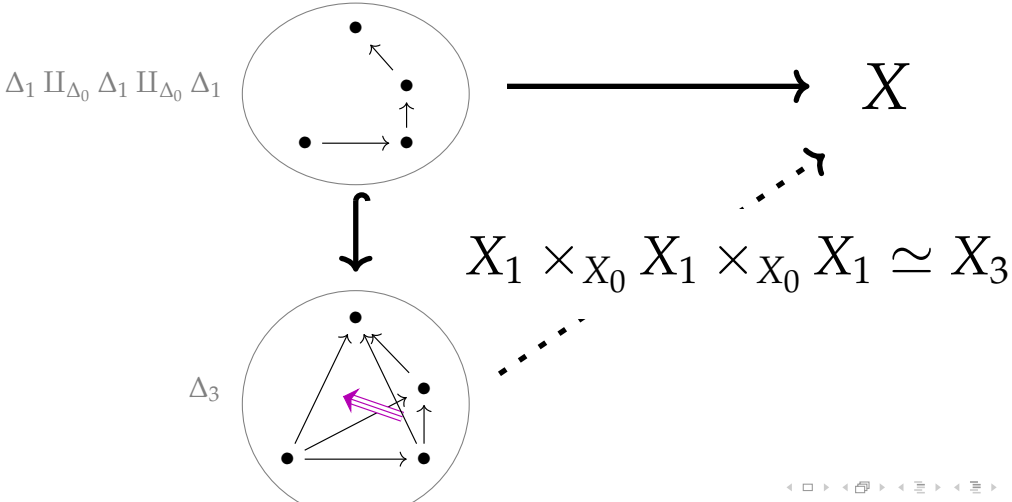
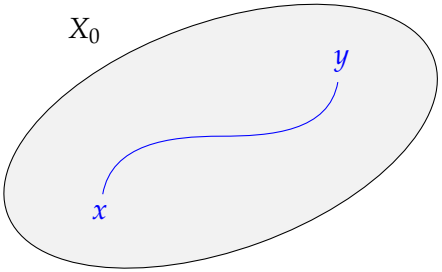


Complete segal spaces

Two coexisting kinds of identifications



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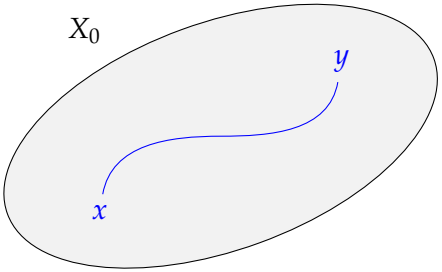

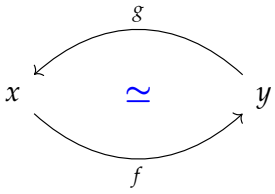

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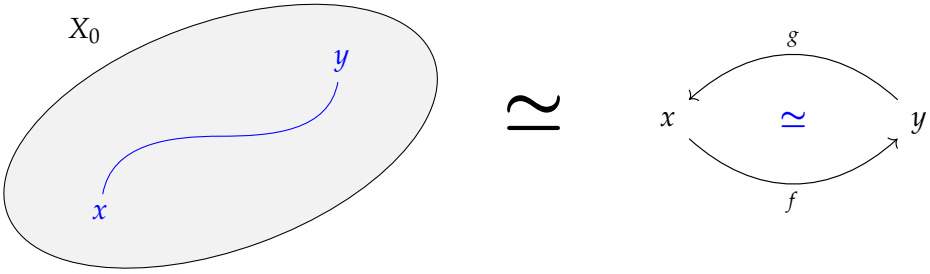

Simplicial type theory

- Riehl, Shulman (2017); Gratzer, Weinberger, Buchholtz…

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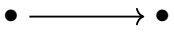
- Goal: $(\infty, 1)$-categories = Complete Segal spaces.

The interval

We add to the syntax a collection of shapes



2

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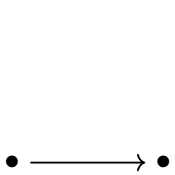

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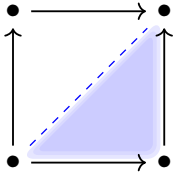

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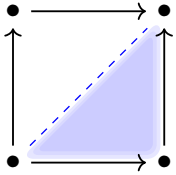

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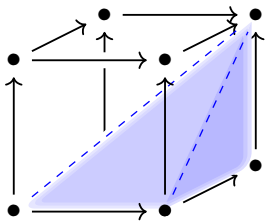

Δ^3

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So each type give rise to a *simplicial type !*

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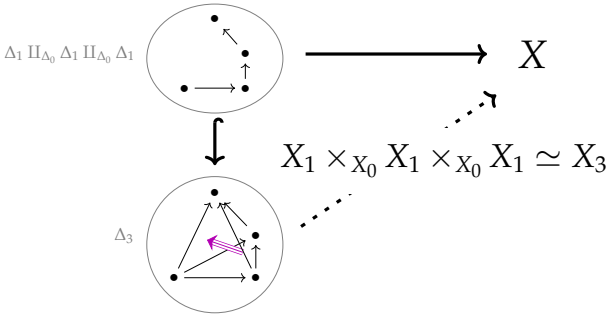

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Reminiscent to Rezk-completes/univalent categories in HoTT.

Hom Types in STT

In STT:

$$\mathsf{hom}_A(x,y) = \sum_{f : \mathsf{I} \to A} (f\,0 = x) \times (f\,1 = y)$$

$$x \text{ ———— } y$$

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$$x \longrightarrow y$$

Then:

$$\mathsf{hom}_{\mathsf{hom}_A(x,y)}(f,g) = \sum_{H : \mathsf{I}^2 \to A} \cdots$$

$$\begin{array}{ccc} x & \xrightarrow{\quad f \quad} & y \\ \| & & \| \\ x & \xrightarrow{\quad g \quad} & y \end{array}$$

The problem of hom-types

In I^2: x, y invertible $\Rightarrow (x, y)$ too.

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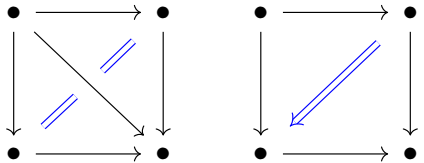

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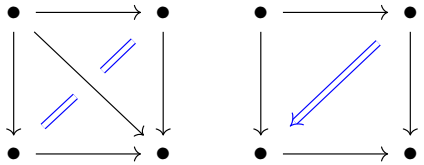
$$\mathsf{hom}_A(x, y) \neq \textstyle\sum_{f : \mathsf{I} \to A} (f\,0 = x) \times (f\,1 = y)$$



$$\mathsf{I} \times \mathsf{I} \qquad \neq \qquad \mathsf{I} \otimes \mathsf{I}$$

Replacing $\to$

- Idea: replacing $(- \times \mathsf{I}) \dashv (\mathsf{I} \to -)$ by $(- \otimes \mathsf{I}) \dashv (\mathsf{I} \multimap -)$

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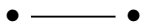

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$\Gamma \otimes \Delta$ $\qquad$ Γ, Δ

$$\Gamma \otimes \Delta \neq \Delta \otimes \Gamma$$

Interpreted as the Crans-Gray tensor product.

Bunches

$$\Gamma, \Delta ::= \diamond \mid x : A \mid \Gamma \otimes \Delta \mid \Gamma, \Delta$$

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crisp asumptions does not care about functoriality.

Rules for $\multimap$ and $\mathrel{-\!\!\circ}$

$$(\multimap\text{-Intro})\ \frac{\Gamma \otimes (x : A) \vdash b : B}{\Gamma \vdash \varrho x.\, b : A \multimap B} \qquad (\mathrel{-\!\!\circ}\text{-Intro})\ \frac{(x : A) \otimes \Gamma \vdash b : B}{\partial x.\, b : A \mathrel{-\!\!\circ} B}$$

$$(\multimap\text{-Elim})\ \frac{\Gamma \vdash f : A \multimap B \qquad \Delta \vdash a : A}{\Gamma \otimes \Delta \vdash f\ |a\rangle : B} \qquad (\mathrel{-\!\!\circ}\text{-Elim})\ \frac{\Gamma \vdash a : A \qquad \Delta \vdash f : A \mathrel{-\!\!\circ} B}{\Gamma \otimes \Delta \vdash \langle a|\, f : B}$$

The $\otimes \dashv (\rightharpoonup, \rightharpoondown)$ adjunctions

For crisp types $X, Y, Z :: \mathcal{U}$

$$(X \otimes Y \rightharpoondown Z) \simeq (X \rightharpoondown Y \rightharpoondown Z) \qquad (X \otimes Y \rightharpoonup Z) \simeq (Y \rightharpoonup X \rightharpoonup Z)$$

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witnessed by currying

$$f \longmapsto \varrho x.\, \varrho y.f\,|\,x \otimes y\rangle \qquad f \longmapsto \partial x.\, \partial y.\, \langle x \otimes y\,|\,f$$

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$$f \longmapsto \varrho x.\, \varrho y.f\, |x \otimes y\rangle \qquad f \longmapsto \partial x.\, \partial y.\, \langle x \otimes y |\, f$$

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$- \otimes Y$ is left adjoint to $Y \rightharpoondown -$

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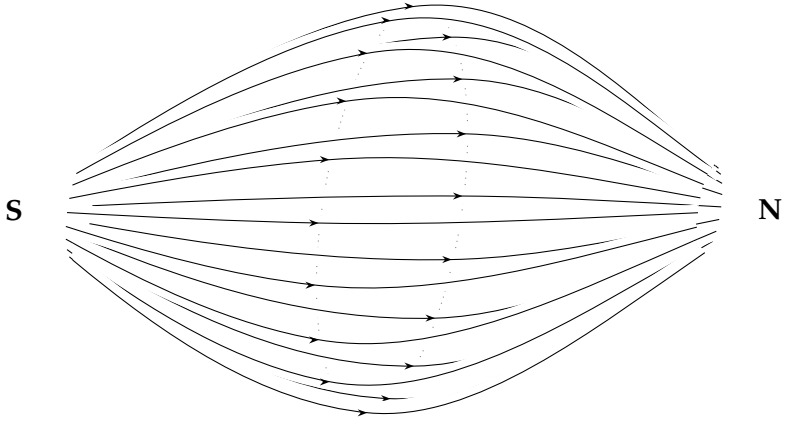
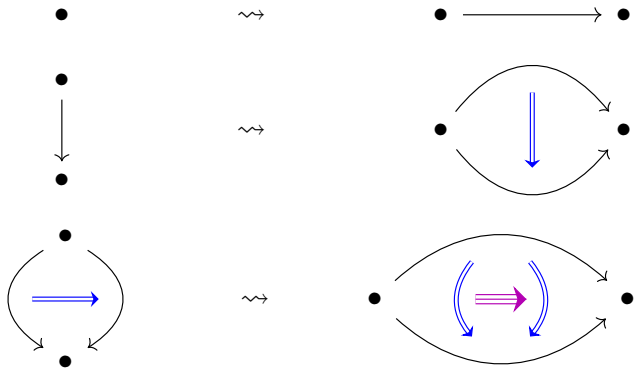
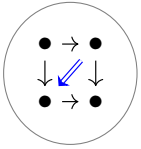
Suspension



Suspension – Examples

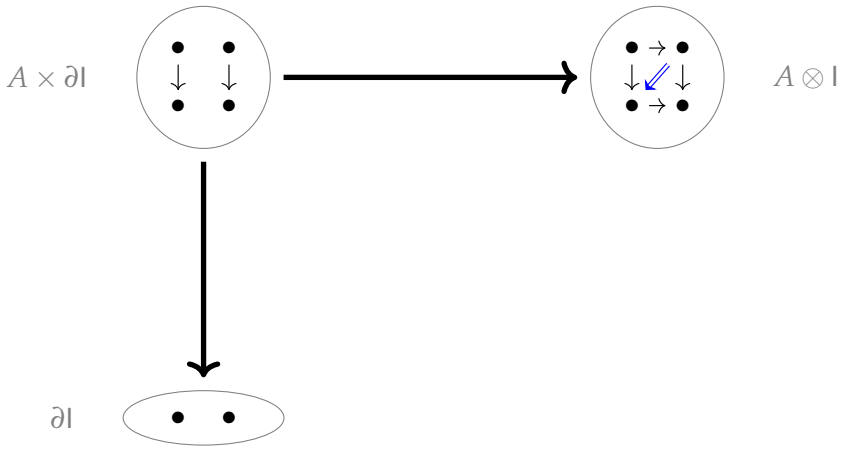


Suspension from $\otimes$



$A \otimes \mathsf{I}$

Suspension from $\otimes$



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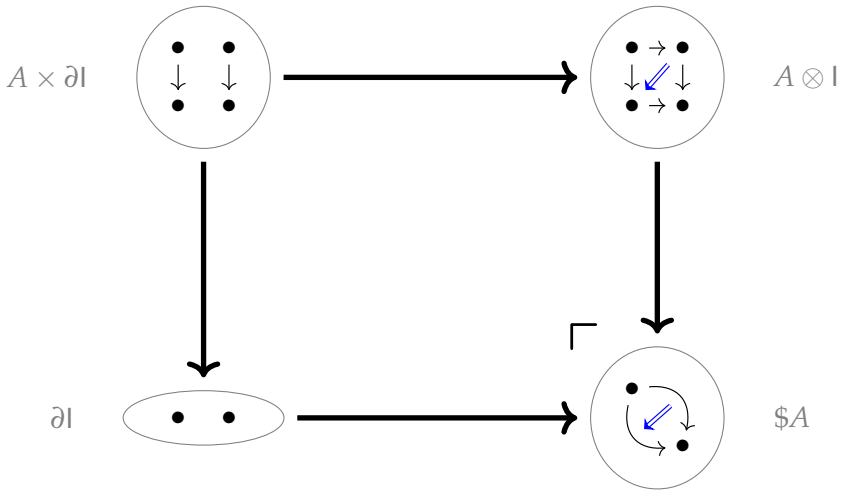

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suspension $\$A$, as a pushout:

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Theorem

$$(\,\$X \to Y \text{ sending } 0, 1 \text{ to } y_0, y_1\,) \;\simeq\; (X \to \mathsf{hom}_Y(y_0, y_1))$$

`batt`: a type-checker for BaTT

/\^._.^/\

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- Small module system, Holes à la Agda, Readable error messages !

```
13) x-11)) (((?13927 x-0) x-1) x-2)))) (λ(x10⊗x11).(Bool_ind(((J(((J(refl) ((?21559 x-2) x-10)) (((postulate54 (λ(x12⊗x13).(x-
2 ⊗ (Bool_ind(postulate36,postulate37) x-13)))) (λ(x12⊗x13).x-13)) (x-10 ⊗ false)))) ((?21576 x-2) x-10)) ((λ(x12,x13).((λ(x14
x15).x-14) x-13)) x-2)),((J(((J(refl) ((?21450 x-2) x-10)) (((postulate54 (λ(x12⊗x13).(x-12 ⊗ (Bool_ind(postulate36,postulate3
) x-13)))) (λ(x12⊗x13).x-13)) (x-10 ⊗ true)))) ((?21458 x-2) x-10)) ((λ(x12,x13).((λ(x14,x15).x-15) x-13)) x-2))) x-11))) fals
) ≡ ((J((x-9 (((postulate53 (λ(x10⊗x11).(x-10 ⊗ (Bool_ind(postulate36,postulate37) x-11)))) (λ(x10⊗x11).x-11)) false))) ((?209
3 x-2) x-9)) ((λ(x10,x11).((λ(x12,x13).x-12) x-11)) x-2))).(((((((postulate100 (λ(x11⊗x12).(x-11 ⊗ (Bool_ind(postulate36,postu
ate37) x-12)))) (λ(x11⊗x12).x-12)) ((λ(x11,x12).x-11) (((?13927 x-0) x-1) x-2))) (λ(x11⊗x12).(((λ(x13,x14).x-13) x-2) (((postu
ate52 (λ(x13⊗x14).(x-13 ⊗ (Bool_ind(postulate36,postulate37) x-14)))) (λ(x13⊗x14).x-14)) (x-11 ⊗ x-12)))))) Bool_ind(((λ(x11,x1
).((λ(x13,x14).x-13) x-12)) (((?13927 x-0) x-1) x-2)),((λ(x11,x12).((λ(x13,x14).x-14) x-12)) (((?13927 x-0) x-1) x-2)))) (λ(x1
⊗x12).(Bool_ind(((J(((J(refl) ((?21189 x-2) x-11)) (((postulate54 (λ(x13⊗x14).(x-13 ⊗ (Bool_ind(postulate36,postulate37) x-14)
)) (λ(x13⊗x14).x-14)) (x-11 ⊗ false)))) ((?21206 x-2) x-11)) ((λ(x13,x14).((λ(x15,x16).x-15) x-14)) x-2)),((J(((J(refl) ((?210
0 x-2) x-11)) (((postulate54 (λ(x13⊗x14).(x-13 ⊗ (Bool_ind(postulate36,postulate37) x-14)))) (λ(x13⊗x14).x-14)) (x-11 ⊗ true))
) ((?21088 x-2) x-11)) ((λ(x13,x14).((λ(x15,x16).x-16) x-14)) x-2))) x-12))) true) ≡ ((J((x-9 (((postulate53 (λ(x11⊗x12).(x-11
⊗ (Bool_ind(postulate36,postulate37) x-12)))) (λ(x11⊗x12).x-12)) true))) (((?20967 x-2) x-9) x-10)) ((λ(x11,x12).((λ(x13,x14).
-14) x-12)) x-2))))
HOLE in file ./hom.batt line 84 characters 1-2 : (Σ(x9 : (x9 : (?13995 x-2)) → (((((((postulate98 (λ(x10⊗x11).(x-10 ⊗ (Bool_in
(postulate36,postulate37) x-11)))) (λ(x10⊗x11).x-11)) ((λ(x10,x11).x-10) (((?13927 x-0) x-1) x-2))) (λ(x10⊗x11).(((λ(x12,x13).
-12) x-2) (((postulate52 (λ(x12⊗x13).(x-12 ⊗ (Bool_ind(postulate36,postulate37) x-13)))) (λ(x12⊗x13).x-13)) (x-10 ⊗ x-11)))))
ool_ind(((λ(x10,x11).((λ(x12,x13).x-12) x-11)) (((?13927 x-0) x-1) x-2)),((λ(x10,x11).((λ(x12,x13).x-13) x-11)) (((?13927 x-0)
x-1) x-2)))) (λ(x10⊗x11).(Bool_ind(((J(((J(refl) ((?23045 x-2) x-10)) (((postulate54 (λ(x12⊗x13).(x-12 ⊗ (Bool_ind(postulate36
postulate37) x-13)))) (λ(x12⊗x13).x-13)) (x-10 ⊗ false)))) ((?23062 x-2) x-10)) ((λ(x12,x13).((λ(x14,x15).x-14) x-13)) x-2)),(
J(((J(refl) ((?22936 x-2) x-10)) (((postulate54 (λ(x12⊗x13).(x-12 ⊗ (Bool_ind(postulate36,postulate37) x-13)))) (λ(x12⊗x13).x-
3)) (x-10 ⊗ true)))) ((?22944 x-2) x-10)) ((λ(x12,x13).((λ(x14,x15).x-15) x-13)) x-2))) x-11))) x-9) ≡ (((λ(x10,x11).x-10) x-2
x-9)).(Σ(x10 : ((((((((postulate100 (λ(x10⊗x11).(x-10 ⊗ (Bool_ind(postulate36,postulate37) x-11)))) (λ(x10⊗x11).x-11)) ((λ(x10
x11).x-10) (((?13927 x-0) x-1) x-2))) (λ(x10⊗x11).(((λ(x12,x13).x-12) x-2) (((postulate52 (λ(x12⊗x13).(x-12 ⊗ (Bool_ind(postul
te36,postulate37) x-13)))) (λ(x12⊗x13).x-13)) (x-10 ⊗ x-11))))) Bool_ind(((λ(x10,x11).((λ(x12,x13).x-12) x-11)) (((?13927 x-0)
x-1) x-2)),((λ(x10,x11).((λ(x12,x13).x-13) x-11)) (((?13927 x-0) x-1) x-2)))) (λ(x10⊗x11).(Bool_ind(((J(((J(refl) ((?22676 x-2
x-10)) (((postulate54 (λ(x12⊗x13).(x-12 ⊗ (Bool_ind(postulate36,postulate37) x-13)))) (λ(x12⊗x13).x-13)) (x-10 ⊗ false)))) ((
22693 x-2) x-10)) ((λ(x12,x13).((λ(x14,x15).x-14) x-13)) x-2)),((J(((J(refl) ((?22567 x-2) x-10)) (((postulate54 (λ(x12⊗x13).(
-12 ⊗ (Bool_ind(postulate36,postulate37) x-13)))) (λ(x12⊗x13).x-13)) (x-10 ⊗ true)))) ((?22575 x-2) x-10)) ((λ(x12,x13).((λ(x1
```

Examples

smimram.github.io/batt/

The end

Thank you !

Questions ?