A Bunched Approach to Directed HoTT

Louise Leclerc

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Outline

Presentation

(Homotopy) Type Theory:

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- Types behaves as spaces.
- Synthetic tool for higher algebra, homotopy theory...
- Suitable for formalizing mathematics / homotopy theory
- **Goal:** replace spaces by *directed spaces*

Intuitionistic Type Theory

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- Basic types: $\mathbb{0}$, $\mathbb{1}$, Bool, $\mathbb{N}$, ...
- Universes $\mathcal{U}$ closed under $\to$, $\times$, $+$, ...
- $\mathbb{0} : \mathcal{U}$, $\mathbb{N} \times \mathbb{1} : \mathcal{U}$, $\mathbb{N} \to \mathsf{Bool} : \mathcal{U}$...

More type formers

- Π-types:

$$A : \mathcal{U},\ B : A \to \mathcal{U} \quad \rightsquigarrow \quad \prod_{x:A} B(x) : \mathcal{U}.$$

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- Σ-types:

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$$a : A,\ u : B(a) \quad \rightsquigarrow \quad (a, u) : \sum_{x:A} B(x).$$

Curry–Howard

Correspondance between types and logical propositions.

- $\to$ is the implication arrow.
- $\times$ is the conjunction.
- $+$ is a constructive disjunction.
- $\mathbb{0}$ is false, $\mathbb{1}$ is true.

$$\text{Logic} \hookrightarrow \text{Algebra}$$

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- In ordinary maths, $=$ is a proposition. Now it becomes a *type*:

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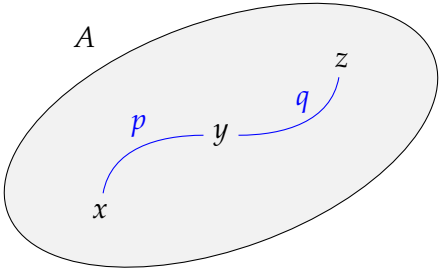
- Think of $x = y$ as the type of paths from x to y.

Groupoid structure on types

- **concatenation**: $p : x =_A y,\ q : y =_A z \ \rightsquigarrow\ p \cdot q : x =_A z$



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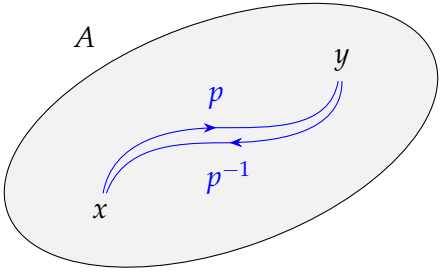

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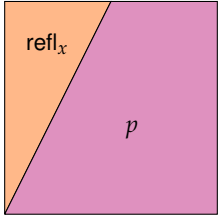

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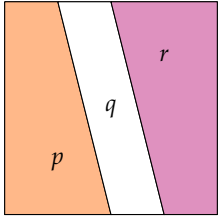

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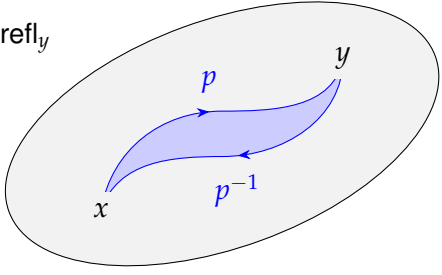

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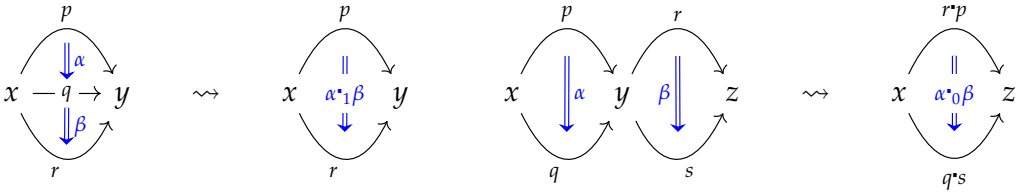
- **composition of higher cells**: $\alpha : p =_{x=y} q, \beta : q =_{x=y} r \rightsquigarrow \alpha \cdot \beta : p =_{x=y} r \ldots$



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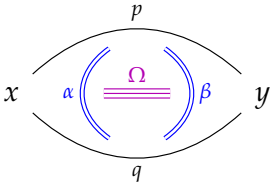
Theorem

Every type is equipped with a canonical structure of ∞-groupoid.

∞-groupoids

Spaces up to homotopy



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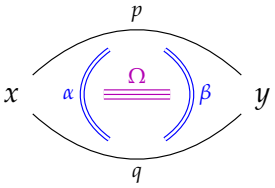

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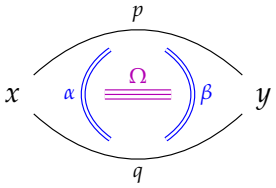

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Formalisation of homotopy theory
(Feudenthal, long exact sequences, sphere homotopy groups...).

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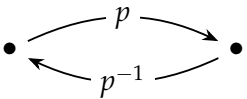
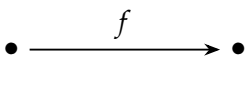


$x = y$



$\mathsf{hom}_A(x, y)$

∞-categories

"Directed" spaces up to homotopy

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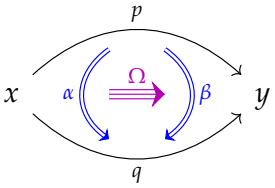

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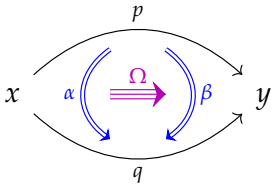

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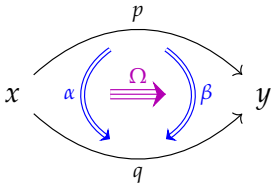

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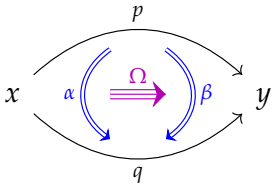

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Rewriting, concurrency,
synthetic manipulation of higher categories.

Simplicial type theory

- Riehl, Shulman (2017); Gratzer, Weinberger, Buchholtz…

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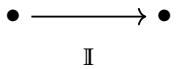
- Goal: define $(\infty, 1)$-categories internally.

The interval

We add to the syntax an interval $(\mathbb{I}, 0, 1, \leq)$



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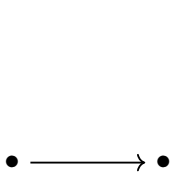
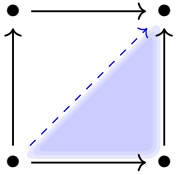


$\mathbb{I}$



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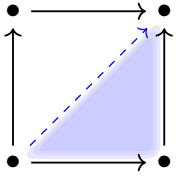

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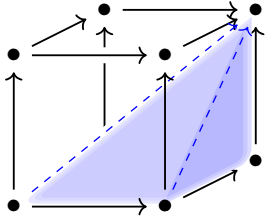

Δ^3

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So each type give rise to a *simplicial type*.

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When does X behaves as a category ?

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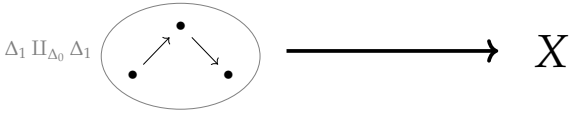

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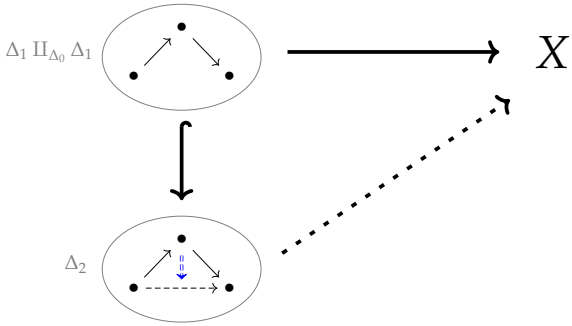

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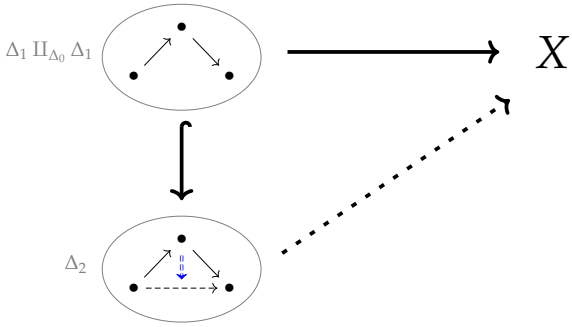

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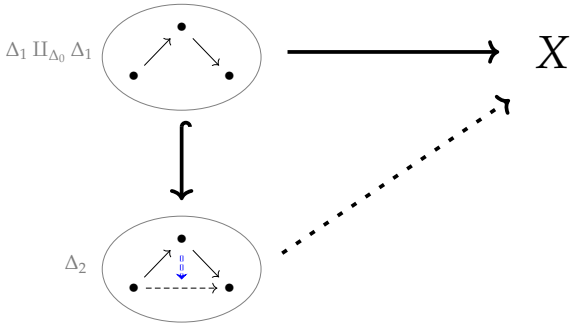

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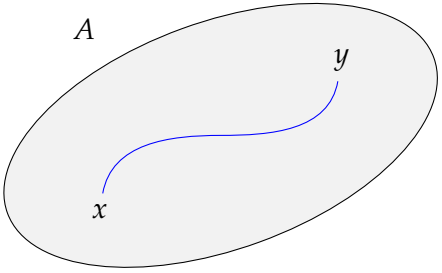
$$\sum_{x,y,z:X} \mathsf{hom}_X(x,y) \times \mathsf{hom}_X(y,z) \xrightarrow{\circ} \mathsf{hom}_X(x,z)$$

Categories (2)

A type A as before admits composites $g \circ f$.



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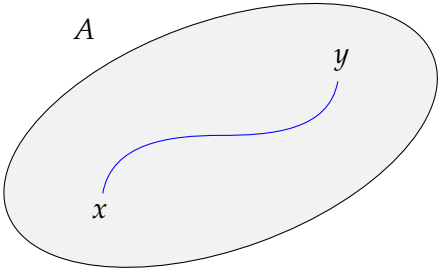

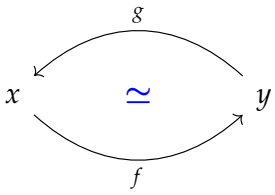

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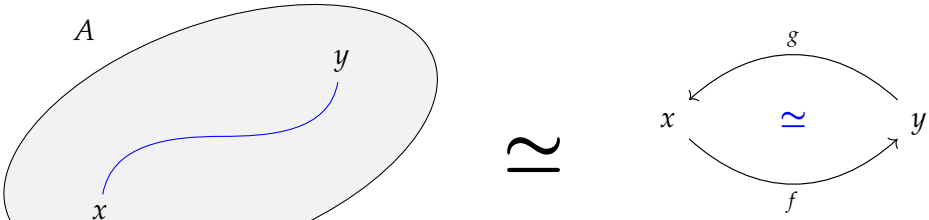
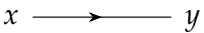


$$(x =_A y) \simeq \mathsf{iso}_A(x, y)$$

Hom Types in STT

Type of 1-cells $= \mathbb{I} \to A$.



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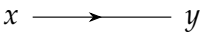

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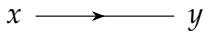
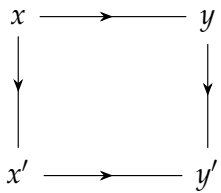


Cells mediating between 1-cells should be 2-cells.
Type of 2-cells $= \mathbb{I} \to (\mathbb{I} \to A) \simeq \mathbb{I}^2 \to A$.



The problem of hom-types

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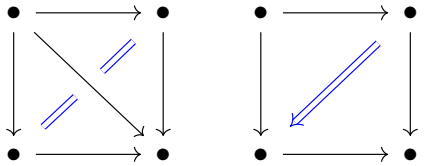

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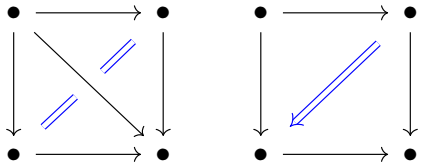

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Γ, Δ

$$\Gamma \otimes \Delta \neq \Delta \otimes \Gamma$$

Interpreted as the Crans-Gray tensor product.

Bunches

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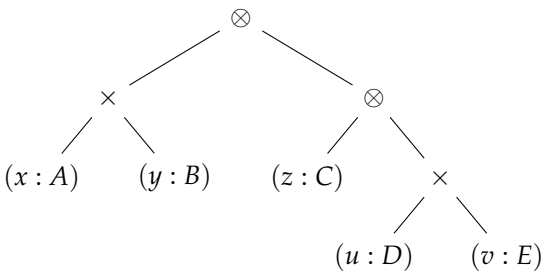

$$\vdash \quad y \otimes (u, v) : B \bigotimes (D \times E)$$

Some rules

$$(\times\text{-Intro})\ \frac{\Gamma \vdash a : A \qquad \Delta \vdash b : B}{\Gamma, \Delta \vdash a, b : A \times B}$$

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The $\otimes \dashv (\rightharpoonup, \rightharpoondown)$ adjunctions

For crisp types $X, Y, Z :: \mathcal{U}$

$$(X \otimes Y \rightharpoondown Z) \simeq (X \rightharpoondown Y \rightharpoondown Z) \qquad (X \otimes Y \rightharpoonup Z) \simeq (Y \rightharpoonup X \rightharpoonup Z)$$

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witnessed by currying

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$- \otimes Y$ is left adjoint to $Y \rightharpoondown -$

$Y \otimes -$ is left adjoint to $Y \rightharpoonup -$

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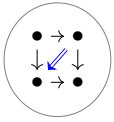

$\mathbb{I} \otimes \mathbb{I}$

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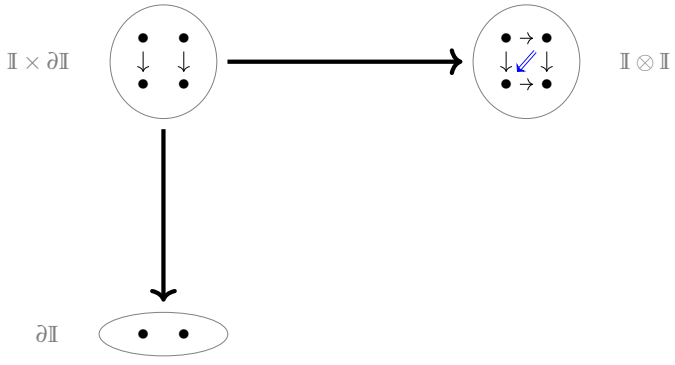

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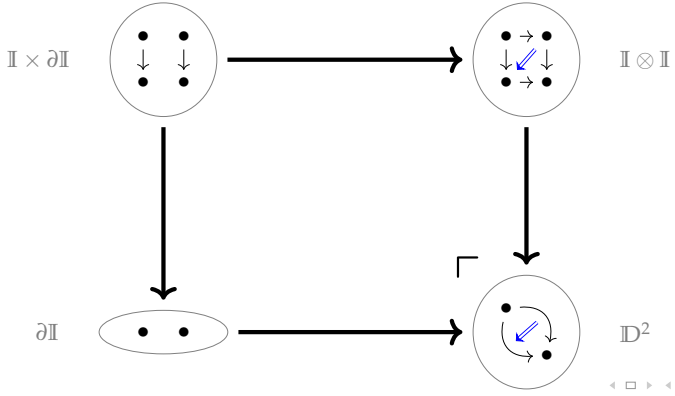

Categories

When does X behaves as a category ?

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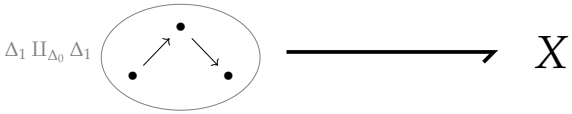

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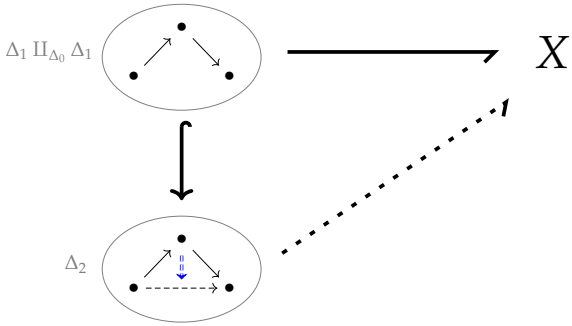

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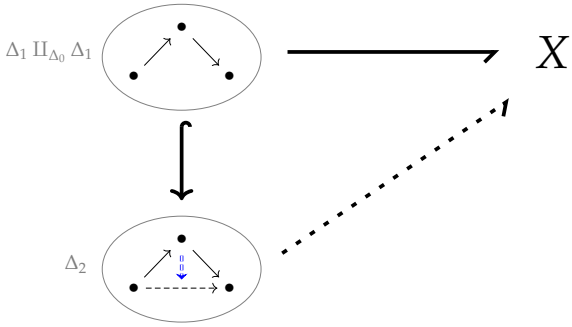

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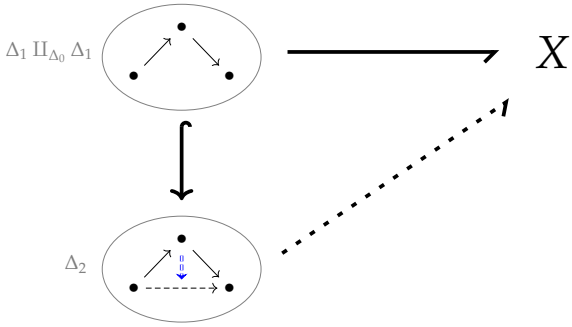

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$$\sum_{x,y,z:X} \mathsf{hom}_X(x,y) \times \mathsf{hom}_X(y,z) \xrightarrow{\circ} \mathsf{hom}_X(x,z)$$

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/\^._.^/\

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- Small module system, Holes à la Agda, Readable error messages !

```
13) x-11)) (((?13927 x-0) x-1) x-2)))) (λ(x10⊛x11).(Bool_ind(((J(((J(refl) ((?21559 x-2) x-10)) (((postulate54 (λ(x12⊛x13).(x-
2 ⊛ (Bool_ind(postulate36,postulate37) x-13)))) (λ(x12⊛x13).x-13)) (x-10 ⊛ false)))) ((?21576 x-2) x-10)) ((λ(x12,x13).((λ(x14
x15).x-14) x-13)) x-2)),((J(((J(refl) ((?21450 x-2) x-10)) (((postulate54 (λ(x12⊛x13).(x-12 ⊛ (Bool_ind(postulate36,postulate3
) x-13)))) (λ(x12⊛x13).x-13)) (x-10 ⊛ true)))) ((?21458 x-2) x-10)) ((λ(x12,x13).((λ(x14,x15).x-15) x-13)) x-2))) x-11))) fals
) ≡ ((J((x-9 (((postulate53 (λ(x10⊛x11).(x-10 ⊛ (Bool_ind(postulate36,postulate37) x-11)))) (λ(x10⊛x11).x-11)) false))) ((?209
3 x-2) x-9)) ((λ(x10,x11).((λ(x12,x13).x-12) x-11)) x-2))).(((((((postulate100 (λ(x11⊛x12).(x-11 ⊛ (Bool_ind(postulate36,postu
ate37) x-12)))) (λ(x11⊛x12).x-12)) ((λ(x11,x12).x-11) (((?13927 x-0) x-1) x-2))) (λ(x11⊛x12).(((λ(x13,x14).x-13) x-2) (((postu
ate52 (λ(x13⊛x14).(x-13 ⊛ (Bool_ind(postulate36,postulate37) x-14)))) (λ(x13⊛x14).x-14)) (x-11 ⊛ x-12))))) Bool_ind(((λ(x11,x1
).((λ(x13,x14).x-13) x-12)) (((?13927 x-0) x-1) x-2)),((λ(x11,x12).((λ(x13,x14).x-14) x-12)) (((?13927 x-0) x-1) x-2)))) (λ(x1
⊛x12).(Bool_ind(((J(((J(refl) ((?21189 x-2) x-11)) (((postulate54 (λ(x13⊛x14).(x-13 ⊛ (Bool_ind(postulate36,postulate37) x-14)
)) (λ(x13⊛x14).x-14)) (x-11 ⊛ false)))) ((?21206 x-2) x-11)) ((λ(x13,x14).((λ(x15,x16).x-15) x-14)) x-2)),((J(((J(refl) ((?210
0 x-2) x-11)) (((postulate54 (λ(x13⊛x14).(x-13 ⊛ (Bool_ind(postulate36,postulate37) x-14)))) (λ(x13⊛x14).x-14)) (x-11 ⊛ true))
) ((?21088 x-2) x-11)) ((λ(x13,x14).((λ(x15,x16).x-16) x-14)) x-2))) x-12))) true) ≡ ((J((x-9 (((postulate53 (λ(x11⊛x12).(x-11
⊛ (Bool_ind(postulate36,postulate37) x-12)))) (λ(x11⊛x12).x-12)) true))) (((?20967 x-2) x-9) x-10)) ((λ(x11,x12).((λ(x13,x14).
-14) x-12)) x-2))))
HOLE in file ./hom.batt line 84 characters 1-2 : (Σ(x9 : (x9 : (?13995 x-2)) → (((((((postulate98 (λ(x10⊛x11).(x-10 ⊛ (Bool_in
(postulate36,postulate37) x-11)))) (λ(x10⊛x11).x-11)) ((λ(x10,x11).x-10) (((?13927 x-0) x-1) x-2))) (λ(x10⊛x11).(((λ(x12,x13).
-12) x-2) (((postulate52 (λ(x12⊛x13).(x-12 ⊛ (Bool_ind(postulate36,postulate37) x-13)))) (λ(x12⊛x13).x-13)) (x-10 ⊛ x-11)))))
ool_ind(((λ(x10,x11).((λ(x12,x13).x-12) x-11)) (((?13927 x-0) x-1) x-2)),((λ(x10,x11).((λ(x12,x13).x-13) x-11)) (((?13927 x-0)
x-1) x-2)))) (λ(x10⊛x11).(Bool_ind(((J(((J(refl) ((?23045 x-2) x-10)) (((postulate54 (λ(x12⊛x13).(x-12 ⊛ (Bool_ind(postulate36
postulate37) x-13)))) (λ(x12⊛x13).x-13)) (x-10 ⊛ false)))) ((?23062 x-2) x-10)) ((λ(x12,x13).((λ(x14,x15).x-14) x-13)) x-2)),(
J(((J(refl) ((?22936 x-2) x-10)) (((postulate54 (λ(x12⊛x13).(x-12 ⊛ (Bool_ind(postulate36,postulate37) x-13)))) (λ(x12⊛x13).x-
3)) (x-10 ⊛ true)))) ((?22944 x-2) x-10)) ((λ(x12,x13).((λ(x14,x15).x-15) x-13)) x-2))) x-11))) x-9) ≡ (((λ(x10,x11).x-10) x-2
x-9)).(Σ(x10 : ((((((((postulate100 (λ(x10⊛x11).(x-10 ⊛ (Bool_ind(postulate36,postulate37) x-11)))) (λ(x10⊛x11).x-11)) ((λ(x10
x11).x-10) (((?13927 x-0) x-1) x-2))) (λ(x10⊛x11).(((λ(x12,x13).x-12) x-2) (((postulate52 (λ(x12⊛x13).(x-12 ⊛ (Bool_ind(postul
te36,postulate37) x-13)))) (λ(x12⊛x13).x-13)) (x-10 ⊛ x-11))))) Bool_ind(((λ(x10,x11).((λ(x12,x13).x-12) x-11)) (((?13927 x-0)
x-1) x-2)),((λ(x10,x11).((λ(x12,x13).x-13) x-11)) (((?13927 x-0) x-1) x-2)))) (λ(x10⊛x11).(Bool_ind(((J(((J(refl) ((?22676 x-2
x-10)) (((postulate54 (λ(x12⊛x13).(x-12 ⊛ (Bool_ind(postulate36,postulate37) x-13)))) (λ(x12⊛x13).x-13)) (x-10 ⊛ false)))) ((
22693 x-2) x-10)) ((λ(x12,x13).((λ(x14,x15).x-14) x-13)) x-2)),((J(((J(refl) ((?22567 x-2) x-10)) (((postulate54 (λ(x12⊛x13).(
-12 ⊛ (Bool_ind(postulate36,postulate37) x-13)))) (λ(x12⊛x13).x-13)) (x-10 ⊛ true)))) ((?22575 x-2) x-10)) ((λ(x12,x13).((λ(x1
```

Examples

smimram.github.io/batt/

The end

Thank you !

Questions ?