# The Fundamental Group

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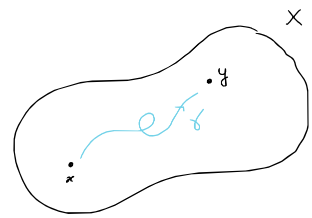
*Dedicated to Aube, in the hope of appealing to her mathematical sensibilities.*

This document is aimed to be an introductive assignement, exploring the fundamental group of a space and Brouwer's fixed point theorem.

## Part 1: The fundamental group of a space

**Definition 1.1 : Path, Loop**

Let $X$ be a topological space. A *path* in $X$ is a continuous map $[0, 1] \to X$. If $a, b$ are points of $X$, a path from $x$ to $y$ is a path $\gamma : [0, 1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$, which we will sometimes write $\gamma : x \to y$. A *loop* in $X$ based at $x$ is a path in $X$ from $x$ to itself. The space $X$ is said to be *path connected* if for any two points $x, y$ of $X$, there is a path from $x$ to $y$.




1) Let $n \in \mathbb{N}$.
   a) Show that $\mathbb{R}^n$ is path connected.
   b) Show that $\mathbb{R}^n \setminus \{0\}$ is path connected when $n \neq 1$.
   c) Let $m \in \mathbb{N}$ and $a_1, \cdots, a_m \in \mathbb{R}^n$, show that $\mathbb{R}^n \setminus \{a_1, \cdots, a_m\}$ is path connected when $n \neq 1$.
2) a) Show that $\mathbb{S}^1 := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ is path connected.
   b) Show that $]0, 1[$ is path connected.
   c) Show that $]0, 1[ \setminus \left\{\frac{1}{2}\right\}$ is not path connected.

**Definition 1.2 : Homeomorphism**

Let $X, Y$ be two topological spaces. A *homeomorphism* $f : X \to Y$ is an invertible continuous map such that its inverse $f^{-1} : Y \to X$ is also continuous. We say that $X$ and $Y$ are *homeomorphic* whenever there is such a homeomorphism $f : X \to Y$ and we write it $X \cong Y$. Hence $\cong$ is an equivalence relation on spaces (if not convinced, check it!)

3) Show that the map $e : ]0, 1[ \to \mathbb{S}^1 \setminus \{(1,0)\}$ defined by $e(t) := (\cos(2\pi t), \sin(2\pi t))$ is a homeomorphism.
4) Let $X \cong Y$ be two homeomorphic spaces, show that $X$ is path connected iff $Y$ is.
5) Show that if $f : X \to Y$ is a homeomorphism, then for any $x \in X$ it induces an homeomorphism $X \setminus \{x\} \to Y \setminus \{f(x)\}$.
6) Deduce from previous questions that $\mathbb{S}^1 \setminus \{(1,0)\}$ is path connected and $\mathbb{S}^1 \setminus \{(1, 0), (-1, 0)\}$ is not.
7) Deduce similarly that $\mathbb{S}^1$ is not homeomorphic to any $\mathbb{R}^n$.

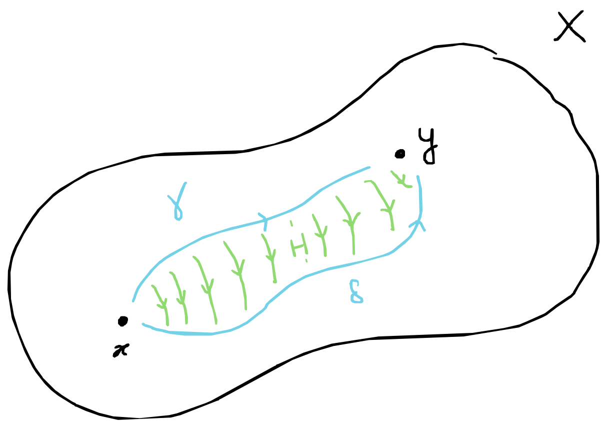
These firsts questions illustrated that the connectedness of a space is an interesting property which is invariant by homeomorphism. Hence it may help to know whether two spaces are homeomorphic or not. Howether this method will become quickly limited in order to compare more complex spaces such as higher spheres. We now define the notion of homotopy, which is to paths what paths are to points, that is, paths between paths.

**Definition 1.3 : Homotopy of paths**

Let $X$ be a space and $\alpha, \beta : [0, 1] \to X$ two paths from $x$ to $y$. A (fixed endpoints) homotopy between $\alpha$ and $\beta$ is the data of a continuous map $H : [0, 1]^2 \to X$ such that

- $\forall t,\ H(0, t) = \alpha(t)$ and $H(1, t) = \beta(t)$
- $\forall s,\ H(s, 0) = x$ and $H(s, 1) = y$

We write $\alpha \sim \beta$ when they are *homotopic*, that is when there is an homotopy between them.

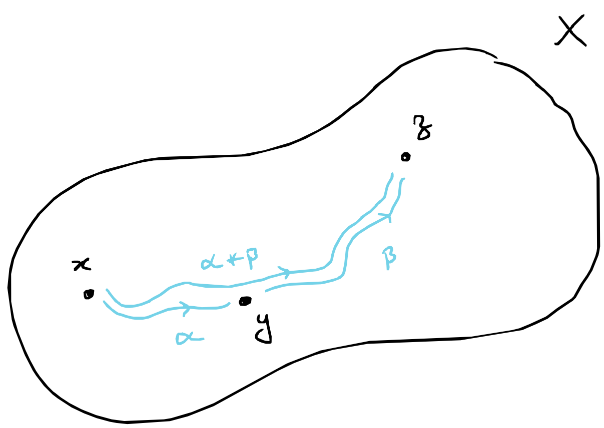



8) Let $X$ be a space and $x, y \in X$, show that $\sim$ induces an equivalence relation on paths of $X$ from $x$ to $y$.

**Definition 1.4**

Let $X$ be a space, let $\alpha : x \to y$ and $\beta : y \to z$ be two paths in $X$. Their concatenation $\alpha * \beta : x \to z$ is the path defined by

$$(\alpha * \beta)(s) = \alpha(2s) \text{ if } s \leq \frac{1}{2} \qquad (\alpha * \beta)(s) = \beta(2s - 1) \text{ if } s \geq \frac{1}{2}$$




9) Let $X$ be a topological space. For any $x \in X$, we denote $c_x$ the constant path $x \to x$.

a) Let $\alpha : x \to y$ be a path. Show that $c_x * \alpha \sim \alpha$ and $\alpha * c_y \sim \alpha$

b) Let $\alpha : w \to x, \beta : x \to y, \gamma : y \to z$ three paths in X. Show that $(\alpha * \beta) * \gamma \sim \alpha * (\beta * \gamma)$.

c) Let $\alpha : x \to y$ be a path, exhibit a path $\beta : y \to x$ such that $\alpha * \beta \sim c_x$ and $\beta * \alpha \sim c_y$.

d) Let $x \in X$. Deduce that $*$ induces a group struture on homotopy classes of loops $x \to x$ in $X$, that is, on $\{\gamma : x \to x\}/_{\sim}$.

**Definition 1.5**

For $X$ a space and $x$ a point of $X$, we denote $\pi_1(X, x)$ the group defined in question 9 $d$. We call it the *fundamental group of X at x*.

10) a) Let $f : X \to Y$ be a continuous map and $x \in X$. Show that the application $\gamma \mapsto f \circ \gamma$ which sends loops at $x$ in $X$ to loops at $f(x)$ in $Y$ induces a group morphism denoted $\pi_1(f, x) : \pi_1(X, x) \to \pi(Y, f(x))$.

b) Show that for $x$ a point of a space $X$, $\pi_1(\mathrm{id}_X, x) = \mathrm{id}_{\pi_1(X,x)}$.

c) Show that for composables continuous maps $X \xrightarrow{f} Y \xrightarrow{g} Z$, and $x \in X$, we have $\pi_1(g \circ f, x) = \pi_1(g, f(x)) \circ \pi_1(f, x)$.

11) Let $f : X \to Y$ be an homeomorphism. Show that $\pi_1(f, x)$ induces a group isomorphism $\pi_1(X, x) \simeq \pi_1(Y, f(x))$.

We now turn to the computation of $\pi_1(\mathbb{S}^1, (1, 0))$. In the following we identify $\mathbb{S}^1$ with the complex unit circle $\mathbb{U} = \{z \in \mathbb{C} \mid |z| = 1\}$, and denote exp the map $s \mapsto e^{2i\pi \cdot s} : \mathbb{R} \to \mathbb{S}^1$. We first aim to show the following lemma:

**Lemma 1.6 : Lifting theorem**

Let $a \leq b \in \mathbb{R}$ and $f : [a, b] \to \mathbb{S}^1$ be a continuous map, there is a continuous map $\hat{f} : [a, b] \to \mathbb{R}$ such that $\exp \circ \hat{f} = f$.

12) a) Let $a \leq b$ and suppose $f : [a, b] \to \mathbb{S}^1$ misses a point (say 1). Show using question 3 that $f$ admits suh a lift $\hat{f} : [a, b] \to \mathbb{R}$.

b) Let $a \leq b$ and consider any continuous map $f : [a, b] \to \mathbb{S}^1$. Show that we may split the interval $[a, b]$ as $a = a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_N = b$ such that for each $i < N$, $f_i := f|_{[a_i, a_{i+1}]} : [a_i, a_{i+1}] \to \mathbb{S}^1$ avoids one point. By 11 $a$, each of them admits a lift $\widehat{f_i} : [a_i, a_{i+1}] \to \mathbb{R}$. Show that we may assume for each $i < N - 1$, $\widehat{f_i}(a_{i+1}) = \widehat{f_{i+1}}(a_{i+1})$. Conclude that $f$ admits a lift $\hat{f}$ which achieve the proof of Lemma 1.6.

We then admit the following variant of Lemma 1.6 which may be proven similarly by cutting $[0, 1]^2$ in little squares.

**Lemma 1.7 : Lifting theorem 2**

Let $f : [0, 1]^2 \to \mathbb{S}^1$ be a continuous map, there is a continuous map $\hat{f} : [0, 1]^2 \to \mathbb{R}$ such that $\exp \circ \hat{f} = f$.

13) We now compute $\pi_1(\mathbb{S}^1, 1)$.

a) Let $\gamma : 1 \to 1$ be a loop in $\mathbb{S}^1$ (seen as $\mathbb{U} \subseteq \mathbb{C}$). Using the lifting lemma, we consider a lift $\widehat{\gamma} : [0, 1] \to \mathbb{R}$ of $\gamma$. We then define the *index* of $\gamma$ to be $\mathrm{ind}(\gamma) := \widehat{\gamma}(1) - \widehat{\gamma}(0)$. Show that $\mathrm{ind}\gamma$ does not depend on the choice of the lift $\widehat{\gamma}$.

b) Show that if $\gamma \sim \delta$ are two homotopic loops $1 \to 1$ in $\mathbb{S}^1$, then $\mathrm{ind}(\gamma) = \mathrm{ind}(\delta)$ (use Lemma 1.7).

c) Show that $\mathrm{ind}(c_1) = 0$ and $\mathrm{ind}(\gamma * \delta) = \mathrm{ind}(\gamma) + \mathrm{ind}(\delta)$

d) Deduce from the previous subquestions that ind induces a group morphism still denoted $\mathrm{ind} : \pi_1(\mathbb{S}^1, 1) \to \mathbb{Z}$.

e) Show that ind is surjective.

f) Let $\gamma, \delta$ be such that $\mathrm{ind}(\gamma) = \mathrm{ind}(\delta)$. Show that we may choose $\widehat{\gamma}, \widehat{\delta}$ such that $\widehat{\gamma}(0) = \widehat{\delta}(0) = 0$ and $\widehat{\gamma}(1) = \widehat{\delta}(1)$. Find a homotopy between $\widehat{\gamma}$ and $\widehat{\delta}$. Deduce that $\gamma \sim \delta$.
Deduce that $\mathrm{ind} : \pi_1(\mathbb{S}^1, 1) \to \mathbb{Z}$ is injective.

g) Conclude.

14) Let $n \in \mathbb{N}$, compute $\pi_1(\mathbb{R}^n)$ and conclude again that $\mathbb{S}^1 \not\cong \mathbb{R}^n$.

15) Explain why we expect $\pi_1(S^2) = \{*\}$ where $\mathbb{S}^2$ is the unit sphere in $\mathbb{R}^3$. Admiting it, deduce that $\mathbb{S}^1 \not\cong \mathbb{S}^2$.

**Definition 1.8 : Homotopy of maps**

Let $f, g : X \to Y$ be two continuous maps. A *homotopy* between $f$ and $g$ is the data of a continuous map $H : [0, 1] \times X \to Y$ such that for $x \in X$, $H(0, x) = f(x)$ and $H(1, x) = g(x)$ We write $f \sim g$ when they are *homotopic*, that is when there is an homotopy between them.

**Definition 1.9 : Homotopy equivalence**

Let $f : X \to Y$ be a continuous map. It is said to be a *homotopy equivalence* iff there is a homotopy inverse map $g : Y \to X$ of $f$. That is such that $g \circ f \sim \mathrm{id}_X$ and $f \circ g \sim \mathrm{id}_Y$. We write $X \simeq Y$ when they are homotopy equivalent. That is when there is a homotopy equivalence between them.

16) We show that some of our construction are homotopy invariant. In the following, $X, Y$ are topological spaces.

a) Let $\gamma : x \to y$ be a path in $X$, and let $\overline{\gamma}$ be the inverse of $\gamma$ up to homotopy defined by $\overline{\gamma}(s) = \gamma(1-s)$ Show that the map $\alpha \mapsto \overline{\gamma} * \alpha * \gamma$ induces a group isomorphism $u_\gamma : \pi_1(X, x) \to \pi_1(X, y)$ which only depend on the homotopy class of $\gamma$.

b) Let $f, g : X \to Y$ be two continuous maps and $x$ a point of $X$. Suppose that we have an homotopy $H : X \times [0, 1] \to Y$ between $f$ and $g$. Let $\gamma = t \mapsto H(x, t)$ the induced path $f(x) \to g(x)$. For $s \in [0, 1]$, write $h_s = z \mapsto H(z, s)$ (we have $h_0 = f$ and $h_1 = g$), consider $\gamma_s : \gamma(s) \to y$ the path defined by $\gamma_s(t) := \gamma(s + (1-s)t)$ and its homotopy inverse $\overline{\gamma_s} : t \mapsto \gamma_s(1-t)$.
Then show that

$$s, t \mapsto \left(\overline{\gamma_s} * (h_s \circ \alpha) * \gamma_s\right)(t)$$

yield a (fixed endpoint) homotopy $f \circ \alpha \sim g \circ \alpha$. Then deduce that we have the identity

$$u_\gamma \circ \pi_1(f, x) = \pi_1(g, x)$$

(where we use the notation of question 16.a.).

17) Suppose we have a homotopy equivalence $f : X \to Y$ with a weak inverse $g$. We aim to show that $\pi_1(f, x)$ is a group isomorphism.

a) using 16.*b*., deduce that $\pi_1(g, f(x))$ both admits a left and a right inverse.

b) Deduce that $\pi_1(g, f(x))$ is an isomorphism, then deduce that $\pi_1(f, x)$ is also an isomorphism.

18) Show that for any $n \in \mathbb{N}$, $\mathbb{R}^n$ is homotopy equivalent to a point and not to $\mathbb{S}^1$. Show also that $\mathbb{S}^1$ is not homotopy equivalent to $\mathbb{S}^2$.

## Part 2: Brouwer's fixed point theorem

In this section, we show the following theorem, using the tools defined in Part 1. We denote by $\mathbb{B}^n$ the closed unit ball of $\mathbb{R}^n$, that is $\mathbb{B}^n := \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 \leq 1\}$. Then $\mathbb{S}^n$ is the boundary of $\mathbb{B}^{n+1}$.

**Theorem 2.1 : Fixed point (Brouwer)**

Let $f : \mathbb{B}^2 \to \mathbb{B}^2$ be a map from the unit disk to itself, then $f$ admits a fixed point.

In order to prove this theorem, we will make use of the following lemma.

**Lemma 2.2 : Non retraction of the disk on its boundary**

There is no continuous map $r : \mathbb{B}^2 \to \mathbb{S}^1$ which restricts to the identity on $\mathbb{S}^1$.

1) We prove Lemma 2.2. Consider such a map $r : \mathbb{B}^2 \to \mathbb{S}^1$.

   a) Remark that $r \circ \iota = \mathrm{id}_{\mathbb{S}^1}$, where $\iota : \mathbb{S}^1 \hookrightarrow \mathbb{B}^2$ is the inclusion of the unit circle into the unit disk. Deduce from the first section that $\pi_1(\mathrm{id}_{\mathbb{S}^1}, (1, 0)) = \pi_1(r, (1, 0)) \circ \pi_1(\iota, (1, 0))$ must be the null morphism $\mathbb{Z} \to \mathbb{Z}$.

   b) Deduce an absurdity.

   We then prove Theorem 2.1 by an *evil* contradiction[1]

2) Let $f : \mathbb{B}^2 \to \mathbb{B}^2$ be a continuous map, and assume by contradiction that it does not have any fixed point. Consider the map $r : \mathbb{B}^2 \to \mathbb{S}^1$ defined as follows: for $x \in \mathbb{B}^2$, $r(x)$ is the unique intersection of the ray (half line) $]f(x), x)$ with the unit circle $\mathbb{S}^1$. Formally, $r(x) = f(x) + \lambda(x)(x - f(x))$ for the unique $\lambda(x) > 0$ such that $r(x) \in \mathbb{S}^1$.

   a) Make a drawing illustrating a point $x$, $f(x)$ and $r(x)$ (yes, this is important!).

   b) Show that $\lambda$ is well defined and continuous. Deduce that $r$ is continuous.

   c) conclude using Lemma 2.2.

[1] This requires the excluded middle, which is of very bad logical taste.