Homotopy of Roses

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A floral 28th birthday gift for Aube

This document is a follow-up to the assignment "The Fundamental Group" and explores the fundamental group of roses (or bouquets of circles) through their universal covering.

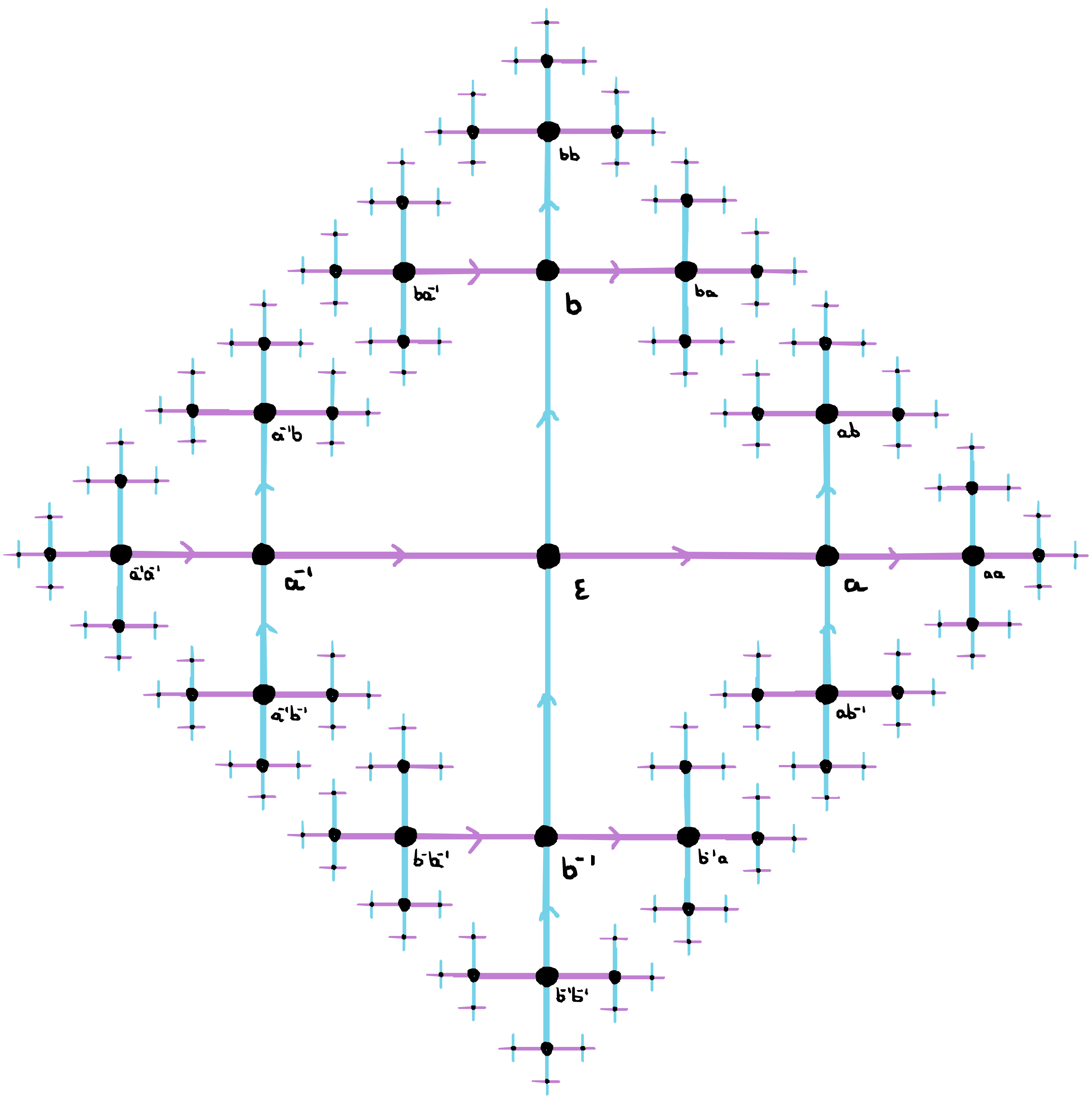

Figure 1: The Cayley graph of the free group on 2 generators.

Part 1: Graphs as combinatorial spaces

In this first section, we give a combinatorial counterpart of homotopy in graphs. Although we will not be exhaustive on this topic, we will explain how to construct a topological space out of a graph, and show some relations between the combinatorial and topological sides.

Definition 1.1 : Graph

In this assignment, a *graph* G is the data of

- a countable set G_0 of *vertices*.
- a countable set G_1 of *edges*.
- two maps src, tgt : $G_1 \to G_0$ respectively assigning to an edge its *source* and *target*.

We let $\widehat{G_1}$ be the set of edges $e \in G_1$ and their *formal inverse* $\overline{e}$. We also define $\overline{\overline{e}} := e$ for $e \in G_1$. We call *signed edges* the elements of $\widehat{G_1}$, and we let $\mathrm{src}(\overline{e}) = \mathrm{tgt}(e)$, $\mathrm{tgt}(\overline{e}) = \mathrm{src}(e)$. A *(non directed) path* in G is an alternating finite sequence $\gamma = (x_0, e_1, x_1, \cdots, x_{n-1}, e_n, x_n)$ of vertices $(x_i)_{0 \le i \le n}$ and signed edges $(e_i)_{1 \le i \le n}$. such that for each $1 \le i \le n$, $\mathrm{src}(e_i) = x_{i-1}$ and $\mathrm{tgt}(e_i) = x_i$. We may write such a path as follow:

$$x_0 \overset{e_1}{—} x_1 \overset{e_2}{—} \cdots \overset{e_n}{—} x_n$$

we may also write $\gamma : x_0 \to x_n$ for emphasizing its endpoints. We write $c_x = (x)$ the only path of length 0 from x to itself. The integer $n \in \mathbb{N}$ is called the length of γ, and we say that γ is a path from x_0 to x_n. The path γ is called *simple* if there is no $1 \le i < n$ such that $e_{i+1} = \overline{e_i}$. The graph G is said to be *connected* if there is at least one path from x to y for each pair of vertices x, y.
It is said to be *acyclic* if there is no simple path of length $n > 0$ from some vertex x to itself in G.
It is said to be a *tree* if it is both connected and acyclic.

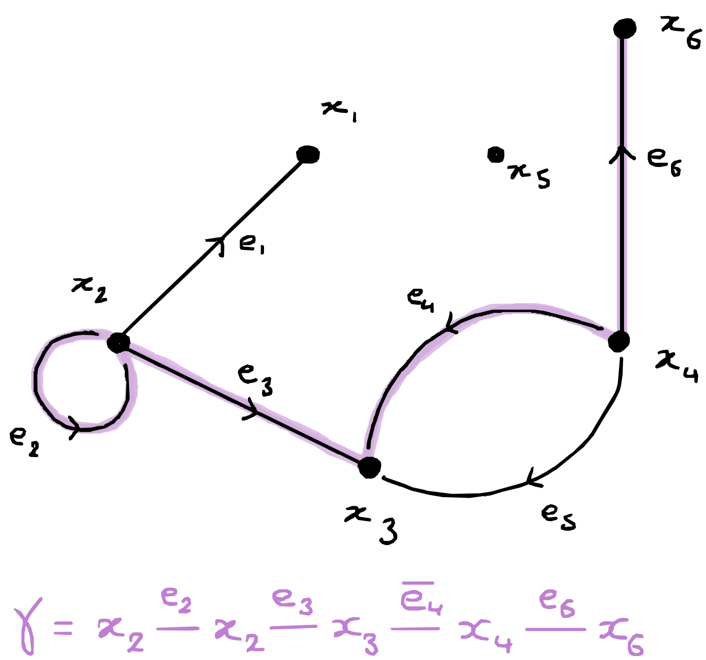

Figure 2: A graph and a path.

Definition 1.2 : concatenation, inverse

Given two paths

$$\gamma = x_0 \overset{e_1}{—} x_1 \overset{e_2}{—} \cdots \overset{e_n}{—} x_n \qquad\qquad \delta = x_n \overset{e_{n+1}}{—} x_{n+1} \overset{e_{n+2}}{—} \cdots \overset{e_m}{—} x_m,$$

in a graph G, we denote $\gamma * \delta$ their *concatenation*, which is defined as the path

$$x_0 \overset{e_1}{—} x_1 \overset{e_2}{—} \cdots \overset{e_n}{—} x_n \overset{e_{n+1}}{—} \cdots \overset{e_m}{—} x_m$$

Given a path γ as above, we denote $\overline{\gamma} := x_n \overset{\overline{e_n}}{—} x_{n-1} \overset{\overline{e_{n-1}}}{—} \cdots \overset{\overline{e_1}}{—} x_0$ which we call its *inverse*.

Definition 1.3 : combinatorial homotopy

Define the *one step reduction* relation $\rightsquigarrow$ on paths by $\gamma \rightsquigarrow \gamma'$ iff $\gamma = x_0 \overset{e_1}{—} x_1 \overset{e_2}{—} \cdots \overset{e_n}{—} x_n$ and $\gamma' = x_0 \overset{e_1}{—} x_1 \overset{e_2}{—} \cdots \overset{e_{i-1}}{—} x_{i-1} = x_{i+1} \overset{e_{i+2}}{—} \cdots \overset{e_n}{—} x_n$. For some $1 \leq i < n$ such that $e_{i+1} = \overline{e_i}$. If $\gamma \rightsquigarrow \gamma'$ we say that γ *reduces to* γ' *in one step.*
We denote by $\rightsquigarrow^*$ its reflexive transitive closure, which we call the *reduction* relation. When $\gamma \rightsquigarrow^* \gamma'$, we say that γ *reduces to* γ'.
Finally, we let $\sim$ denote the symetric closure of $\rightsquigarrow^*$, which we call the *(combinatorial) homotopy* relation. When $\gamma \sim \gamma'$, we say that they are *(combinatorially) homotopic*.

1) Let G be a graph.

   a) Let γ be a path in G such that $\gamma \rightsquigarrow \gamma'$ and $\gamma \rightsquigarrow \gamma''$. Show that either $\gamma' = \gamma''$ or there is a path δ such that $\gamma' \rightsquigarrow \delta$ and $\gamma'' \rightsquigarrow \delta$.
   b) Deduce that for any path γ in G, there is a unique simple path – which we denote $\gamma^{\downarrow}$ – such that $\gamma \rightsquigarrow^* \gamma^{\downarrow}$. We call $\gamma^{\downarrow}$ the *simple path* associated to γ.
   c) Deduce that there is exactly one simple path in each equivalent class of $\sim$.

2) Let G be a graph.

   a) Show that the concatenation operation $*$ is associative, and that for any path $\gamma : x \to y$ in G, $\gamma * \overline{\gamma} \rightsquigarrow^* c_x$ and $\overline{\gamma} * \gamma \rightsquigarrow^* c_y$.
   b) Let $\gamma, \gamma' : x \to y$ and $\delta, \delta' : y \to z$ be paths in G. Show that $\gamma \sim \gamma'$ and $\delta \sim \delta'$ implies $\gamma * \delta \sim \gamma' * \delta'$.
   c) Show that in an acyclic graph (resp. a tree), there is at most (resp. exactly) one simple path between any two vertices.

Definition 1.4 : Realization of a graph

Let $G = (G_0, G_1, \mathsf{src}, \mathsf{tgt})$ be a graph. Its *realization* is the quotient topological space

$$|G| := \left(G_0 \amalg \coprod_{e \in G_1} [0,1]\right) \Big/ \mathcal{G}\mathsf{lue}$$

where $\mathcal{G}\mathsf{lue}$ is the equivalence relation generated by:

- $\forall e \in G_1$, $\mathsf{src}(e)\ \mathcal{G}\mathsf{lue}\ (e, 0)$
- $\forall e \in G_1$, $\mathsf{tgt}(e)\ \mathcal{G}\mathsf{lue}\ (e, 1)$

Let $\langle\cdot\rangle : \left(G_0 \amalg \coprod_{e \in G_1} [0,1]\right) \to |G|$ denote the quotient map. Recall that the *quotient topology* on $|G|$ is the topology generated by the $U \subseteq |G|$ such that $\langle\cdot\rangle^{-1}(U)$ is open. It is equivalently described as the final topology for $\langle\cdot\rangle$, that is the finest topology on $|G|$ which makes $\langle\cdot\rangle$ continuous.

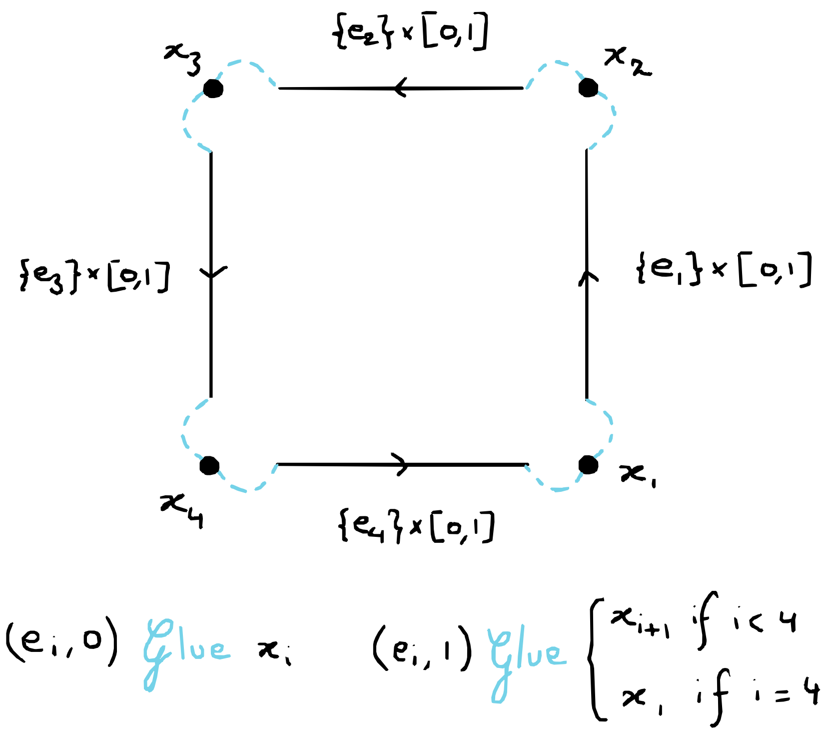

Figure 3: Realization of a square by gluing edges and vertices

3) *(optional, you may admit this result)* Let G be a graph. We define the quotient pseudometric d on $|G|$ by the formula

$$d\big(\langle e,s\rangle,\ \langle e',t\rangle\big) := \inf_{(e_i,s_i),\ (e_i,t_i)\mathcal{G}\mathsf{lue}(e_{i+1},s_i)} \sum_{0\leq i<n} |t_i - s_i|$$

and for x a vertex which is neither a source or a target and χ any class in $|G|$

$$d(\langle x\rangle,\ \chi) = d(\chi,\ \langle x\rangle) := \begin{cases} 0 \text{ if } x \in \chi \\ +\infty \text{ otherwise} \end{cases}$$

Where the infimum ranges over the set of all finite sequences of pairs $(e_i,\ s_i)_{1\leq i\leq n}$ and $(e_i,\ t_i)_{1\leq i\leq n}$ such that $n \geq 1$, $(e_1, s_1)\ \mathcal{G}\mathsf{lue}\ (e,s)$, $(e_n, t_n)\ \mathcal{G}\mathsf{lue}\ (e',t)$ and for each $1 \leq i < n$, $(e_i,\ t_i)\ \mathcal{G}\mathsf{lue}\ (e_{i+1},\ s_i)$.
Show that the quotient pseudometric on $|G|$ is a well-defined (generalized) metric on $|G|$, which induces a coarser topology than the quotient one. In particular, $|G|$ is always Hausdorff. Assume moreover that G is such that for each vertex x of G, $\{e \mid \mathsf{src}(e) = x \vee \mathsf{tgt}(e) = x\}$ is finite, and show that the metric d induces the quotient topology on $|G|$.

Definition 1.5 : Realization of a path

Let G be a graph and $\gamma = x_0 \xrightarrow{e_1} x_1 \xrightarrow{e_2} \cdots \xrightarrow{e_n} x_n$ be a path in G. We define its realization $|\gamma| : [0,1] \to |G|$ as follows:

$$\text{if } \gamma = c_x, \quad |\gamma| := c_{\langle x\rangle}.$$

$$\text{otherwise,} \quad \text{for } 1 \leq i \leq n \text{ and } t \in [(i-1)/n,\ i/n], \quad |\gamma|(t) := \begin{cases} \langle e_i, (n\cdot t - i + 1)\rangle & \text{if } e_i \in G_1 \\ \langle \overline{e_i}, (i - n\cdot t)\rangle & \text{if } e_i \in \widehat{G_1} \setminus G_1 \end{cases}$$

4) Let G be a graph

a) Let γ be a path in G. Show that $|\gamma|$ is a well defined path from $\langle x\rangle$ to $\langle y\rangle$ in $|G|$.

b) For γ, δ two concatenable paths in G, show that $|\gamma * \delta| \sim |\gamma| * |\delta|$.

c) For γ, δ two paths in G such that $\gamma \sim \delta$, show that $|\gamma| \sim |\delta|$.

Definition 1.6 : Homotopy group of a graph

Let G be a graph, and x a vertex of G. We denote $(\Omega(G, x), *)$ the monoid of paths in G from x to itself. Then the *homotopy group of* G is the group $\Omega(G, x)/_{\sim}$ with the law induced by $*$. We denote it $\pi_1(G, x)$. In general we denote $[\gamma]$ the class of any path γ under the relation $\sim$.

5) Let G be a graph and x a vertex of G, we want to show that the realization induces a surjective morphism $\pi_1(G, x) \twoheadrightarrow \pi_1(|G|, \langle x \rangle)$.

   a) Show that $[\gamma] \mapsto [|\gamma|]$ is a well defined application from classes of paths from x to y in G, to classes of paths from $\langle x \rangle$ to $\langle y \rangle$ in $|G|$. And that it is compatible with the operation induced by $*$. In particular, it yields a group morphism $\Phi : \pi_1(G, x) \to \pi_1(|G|, \langle x \rangle)$.

   b) Let $\gamma : \langle x \rangle \to \langle y \rangle$ in $|G|$. Show that $\gamma \sim \gamma_1 * \cdots * \gamma_n$ for some $n \in \mathbb{N}$ (by convention $\gamma \sim c_x$ if $n = 0$) and for each i, $\gamma_i : \langle x_i \rangle \to \langle x_{i+1} \rangle$ coretricts to the image of an edge $\langle \{e_i\} \times [0, 1] \rangle \subseteq |G|$. Conclude that γ is homotopic to $|\delta|$ for some path δ in G. In particular, Φ is surjective.

6) Let G be an acyclic graph and x a vertex of G, show that $\pi_1(|G|, \langle x \rangle)$ is trivial.

Part 2: Roses and regular trees

In this section, we will introduce roses and their universal covering, which are realizations of regular trees, or equivalently of Cayley graphs for free groups. We will then compute the homotopy groups of roses.
We fix a countable family of symbols $(\ell_i)_{i \in \mathbb{N}^*}$ and write $\mathcal{A}_n$ for $\{\ell_i\}_{i \leq n}$.

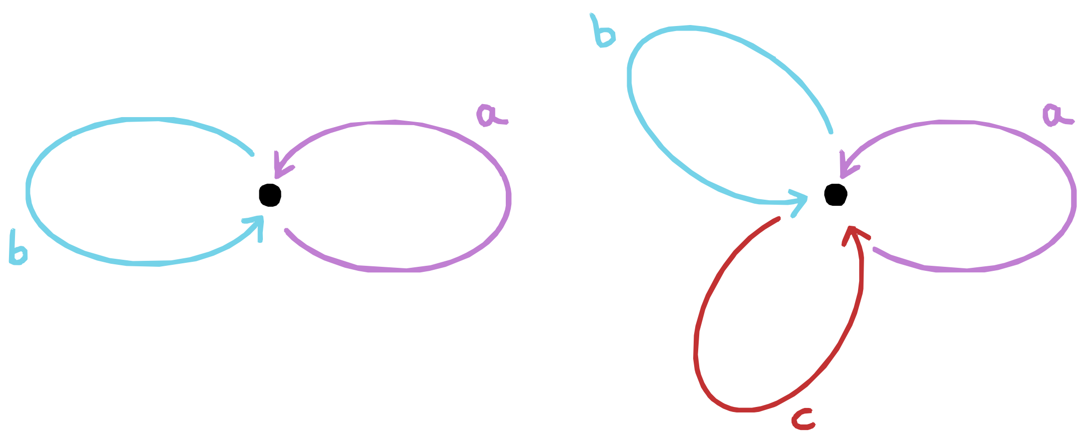

Figure 4: Rose graphs with 2 and 3 petals.

Definition 2.1 : Roses, Free groups

Let $\mathcal{A}$ be any countable set. The *rose graph with* $\mathcal{A}$ *petals* $R_{\mathcal{A}}$ is defined as follows:

- It has only one vertex which we denote $\bullet$.
- It has for edges the set $\mathcal{A}$.

We denote $\mathcal{R}_{\mathcal{A}} := |R_{\mathcal{A}}|$ its realization, which we call the *rose with* $\mathcal{A}$ *petals*. The *free group on* $\mathcal{A}$, denoted $\mathbb{F}_{\mathcal{A}}$ is defined as $\pi_1(R_{\mathcal{A}}, \bullet)$. When seen as a representant of an element of $\mathbb{F}_n$, a path $w : \bullet \to \bullet$ in R_n may be written as a word $e_1 \cdots e_m$ for $e_i \in \widehat{\mathcal{A}}$, it is also called a *reduced word* whenever it is simple.
When $\mathcal{A} = \mathcal{A}_n$, we call $R_n := R_{\mathcal{A}_n}$ the *rose graph with n petals*, $\mathcal{R}_n := \mathcal{R}_{\mathcal{A}_n}$ the *rose with n petals* and $\mathbb{F}_n := \mathbb{F}_{\mathcal{A}_n}$ the *free group on n generators*.

Definition 2.2 : Regular tree

Let $\mathcal{A}$ be any countable set, The $2\mathcal{A}$-*regular tree* $T_\mathcal{A}$ is defined as follows:

- It has for vertices the elements of the free group $\mathbb{F}_\mathcal{A}$.
- For each $w \in \mathbb{F}_\mathcal{A}$ and $e \in \mathcal{A}$, there is an edge (w, e) in $T_\mathcal{A}$ with $\mathtt{src}((w,e)) = w$ and $\mathtt{tgt}((w,e)) = we$.

We write $\mathcal{T}_\mathcal{A}$ for its realization $|T_\mathcal{A}|$. When $\mathcal{A} = \mathcal{A}_n$, we call $T_n := T_{\mathcal{A}_n}$ the $2n$-*regular tree*, and write $\mathcal{T}_n$ for $\mathcal{T}_{\mathcal{A}_n}$. There is a canonical continuous map $p : \mathcal{T}_\mathcal{A} \to \mathcal{R}_\mathcal{A}$. which is defined as follows when $\mathcal{A} \neq \emptyset$:

$$p(\langle (w,e), s\rangle) := \langle e, s\rangle$$

Exemple 2.3

A representation of the 4-regular tree T_2 is given in Figure 1 on the frontpage. Beware that it is a non-isometric illustration of its realization $\mathcal{T}_2$.

In the remaining of this section, we fix a non-empty countable set $\mathcal{A}$ and study the spaces $\mathcal{R}_\mathcal{A}$ and $\mathcal{T}_\mathcal{A}$.

7) Show that p is a well-defined continuous map.

8) We show that $(\mathcal{T}_\mathcal{A}, p)$ is a *covering* of $\mathcal{R}_\mathcal{A}$. More explicitely, let $x \in \mathcal{R}_\mathcal{A}$, we show that there is an open neighborhood of U_x of x such that $p^{-1}(U_x) = \bigsqcup_{w \in \mathbb{F}_\mathcal{A}} V_{x,w}$ and for each $w \in \mathbb{F}_\mathcal{A}$, $p|_{V_{x,w}}^{U_x} : V_x \to U_x$ is a homeomorphism. Refer to Figure 5 for an illustration.

   a) We consider the case $x = \langle e, s\rangle$ for some $e \in \mathcal{A}$ and $s \in]0, 1[$. Show that the following opens are suitable.

   $$U_x = \langle \{e\} \times]0,1[\,\rangle \qquad\qquad V_{x,w} = \left\langle \{(w,e)\} \times]0,1[\,\right\rangle$$

   b) We consider the case $x = \langle \bullet \rangle \in \mathcal{R}_\mathcal{A}$. Show that the following opens are suitable.

   $$U_x = \langle \mathcal{A} \times ([0,1] \setminus \{1/2\})\rangle \qquad V_{x,w} = \left\langle \{(w,e)\}_{e\in\mathcal{A}} \times [0, 1/2[\,\right\rangle \cup \left\langle \{(w\bar{e}, e)\}_{e\in\mathcal{A}} \times]1/2, 1]\right\rangle$$

We now focus on proving the following lemmas, which will allow us to lift paths and homotopies in $\mathcal{R}_\mathcal{A}$ on its covering space $\mathcal{T}_\mathcal{A}$ in a very similar fashion to what we did in the first assignement with the covering $\exp : \mathbb{R} \to \mathbb{S}^1$.

Lemma 2.4 : Path lifting property

Let $t < t' \in \mathbb{R}$ and $\gamma : [t, t'] \to \mathcal{R}_\mathcal{A}$ be a continuous map with $x := \gamma(t)$, and let $y \in p^{-1}(x)$. There exists a unique lift $\widehat{\gamma} : [t, t'] \to \mathcal{T}_\mathcal{A}$ such that $p \circ \widehat{\gamma} = \gamma$ and $\widehat{\gamma}(0) = y$.

Lemma 2.5 : Homotopy lifting property

Let $t < t', s < s' \in \mathbb{R}$ and $H : [s, s'] \times [t, t'] \to \mathcal{R}_\mathcal{A}$ be a continuous map with $\gamma := H(s, \cdot) : [t, t'] \to \mathcal{R}_\mathcal{A}$, and let $\widehat{\gamma} : [t, t'] \to \mathcal{T}_\mathcal{A}$ be a lift of γ, that is with $p \circ \widehat{\gamma} = \gamma$. Then there exists a unique lift $\widehat{H} : [s, s'] \times [t, t'] \to \mathcal{T}_\mathcal{A}$ such that $p \circ \widehat{H} = H$ and $\widehat{H}(s, \cdot) = \widehat{\gamma}$.

9) Show Lemma 2.4.

10) We now focus on Lemma 2.5. Consider $t < t'$, $s < s'$, H, γ and $\widehat{\gamma}$ as in the statement of the Lemma.

    a) First assume that $H([s, s'] \times [t, t']) \subseteq U_x$ for some U_x as in the statement of question 8. Then show Lemma 2.5 in this case.

    b) Now let $u \in [t, t']$, show that we may find an interval $N_u \subseteq [t, t']$ which is a neighborood of u in $[t, t']$, together with a subdivision $s = s_0 < s_1 < \cdots < s_m = s'$ of $[s, s']$ such that for each $0 \leq i < m$, $H([s_i, s_{i+1}] \times N_u) \subseteq U_x$ for some U_x as above.

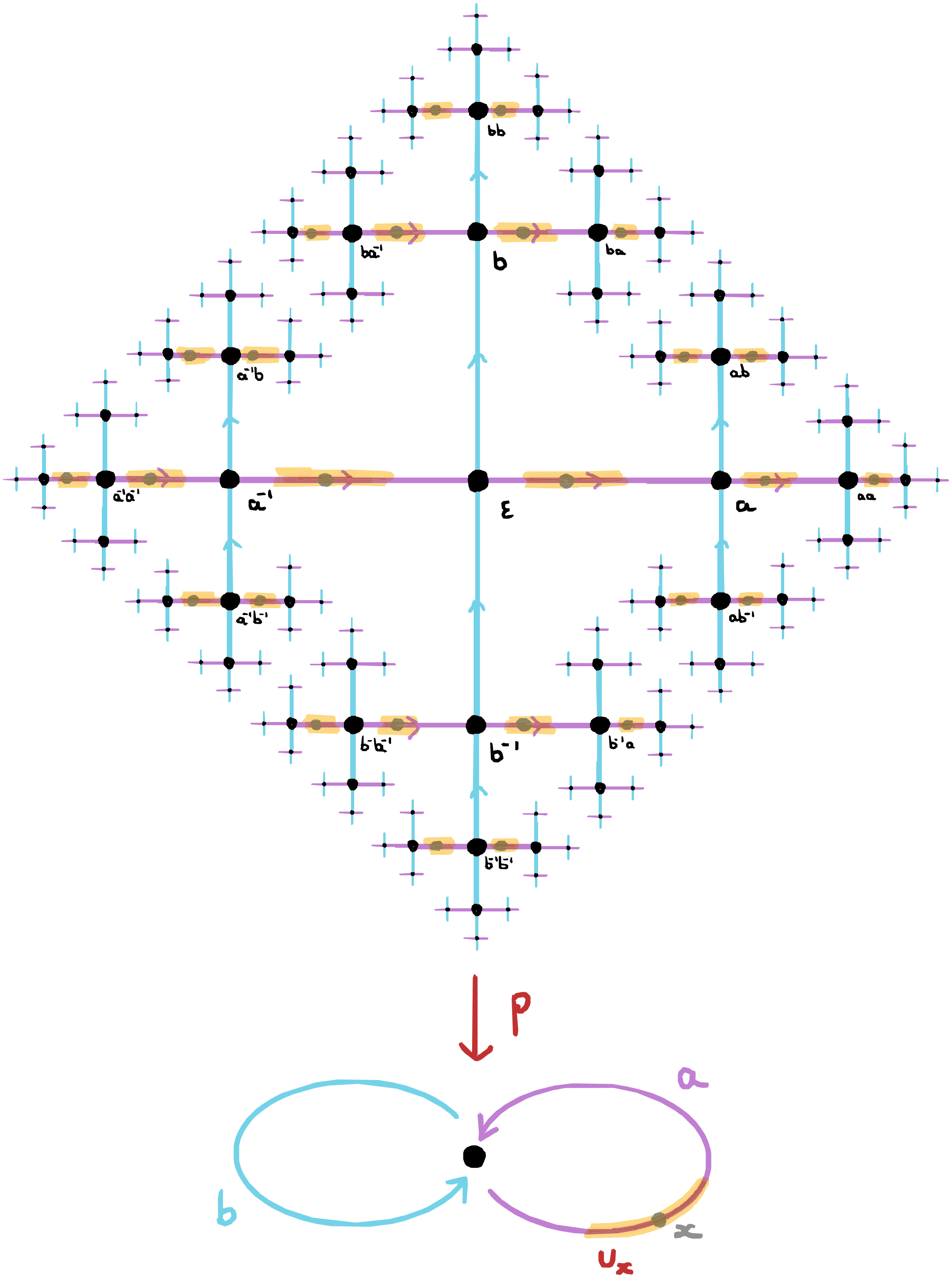

Figure 5: A point x in $\mathcal{R}_2$ and its preimages, the open neighboroods U_x and its preimages.

c) Deduce that for each $u \in [t, t']$, there is an interval $N_u \subseteq [t, t']$ which is a neighborood of u in $[t, t']$ on which the lift $\hat{H}|_{[s,s']\times N_u}$ exists and is unique. (Beware that N_u might need to be smaller than that found in 10.b)

d) Conclude that $\widehat{H}$ exists and is unique.

We make a small detour and define a nice structure on $\mathcal{T}_\mathcal{A}$: a continuous action of $\mathbb{F}_\mathcal{A}$ on the space $\mathcal{T}_\mathcal{A}$, which acts simply transitively on each fibers $p^{-1}(x)$ for any $x \in \mathcal{R}_\mathcal{A}$.

Definition 2.6 : $\mathbb{F}_\mathcal{A}$-action and division in $\mathcal{T}_\mathcal{A}$

We define a $\mathbb{F}_\mathcal{A}$-action on $\mathcal{T}_\mathcal{A}$ by the formula

$$u \cdot \langle (w, e), s\rangle := \langle (uw, e), s\rangle \quad (w, u \in \mathbb{F}_\mathcal{A}, e \in \mathcal{A}, s \in [0, 1]).$$

Given any two points y, z in $\mathcal{T}_\mathcal{A}$ such that $p(y) = p(z)$, we denote by z/y the unique element of $\mathbb{F}_\mathcal{A}$ such that $z/y \cdot y = z$.

11) a) Check that $\cdot$ is well-defined, and is indeed a $\mathbb{F}_\mathcal{A}$-action.

b) Show that the division operation on $\mathcal{T}_\mathcal{A}$ is well defined and continuous $\mathcal{T}_\mathcal{A} \times_p \mathcal{T}_\mathcal{A} \to \mathbb{F}_\mathcal{A}$. Where $\mathcal{T}_\mathcal{A} \times_p \mathcal{T}_\mathcal{A}$ denotes the subspace $\{(y, z) \in \mathcal{T}_\mathcal{A}^2 \mid p(y) = p(z)\}$ of $\mathcal{T}_\mathcal{A}^2$.

We may now finally compute the fundamental group of the rose $\pi_1(\mathcal{R}_\mathcal{A}) := \pi_1(\mathcal{R}_\mathcal{A}, \langle\bullet\rangle)$. The following definition will be useful to address this computation.

Definition 2.7 : Holonomy

Let $\gamma : \langle\bullet\rangle \to \langle\bullet\rangle$ be a loop in $\mathcal{R}_\mathcal{A}$. Let $\widehat{\gamma}$ be a lift of γ as given by Lemma 2.4. We define the *holonomy of* γ by the formula

$$\mathsf{hol}(\gamma) := \widehat{\gamma}(0)^{-1}\widehat{\gamma}(1) \in \mathbb{F}_\mathcal{A}$$

Where we identify $\widehat{\gamma}(0)$ and $\widehat{\gamma}(1)$ with their unique element in $\mathbb{F}_\mathcal{A}$.

12) Show that the holonomy of a path γ is a well defined element of $\mathbb{F}_\mathcal{A}$.

13) Show that hol induces a group isomorphism $\pi_1(\mathcal{R}_\mathcal{A}) \to \mathbb{F}_\mathcal{A}$.

Part 3: Homotopy groups of graphs

In this last section, we come back to graphs and give a method to compute their homotopy groups. We also verify that combinatorial graphs (with a spanning tree) and their realizations have the same notion of homotopy. Finally, as an application, we compute the fundamental group of an infinite square grid.

Definition 3.1 : Subgraph

Let G be a graph. A subgraph $G' = (G'_0, G'_1)$ of G (written $G' \subseteq G$) is the data of

- A subset $G'_0 \subseteq G_0$ of vertices.
- A subset $G'_1 \subseteq G_1$ of edges.

which are closed under the src and tgt operations.

Definition 3.2 : Spanning tree

Let G be a connected graph. A *spanning tree of* G is the data of a subgraph $T \subseteq G$, such that T is a tree and has for vertices all the vertices of G.

14) Let G be a finite[1] connected graph. Show that G admits a spanning tree.

15) Let G be a connected graph and $T \subseteq G$ a spaning tree of G. Let x be a vertex of G. We compute $\pi_1(G, x)$. Let $\mathcal{A} := G_1 \setminus T_1$. For $e \in \mathcal{A}$, we write δ_e for the unique simple path $x \to x$ of the form $\gamma * (y, e, z) * \gamma'$ with γ and γ' simple paths in T. Show that the map $e \mapsto \delta_e : \mathcal{A} \to \Omega(G, x)$ extends to a unique group isomorphism $\Psi : \mathbb{F}_{\mathcal{A}} \to \pi_1(G, x)$.

16) We finally study the homotopy group of a realization of a graph. Let G be a connected graph and $T \subseteq G$ a spanning tree of G. As above, let $\mathcal{A} := G_1 \setminus T_1$

a) We consider the map $\mathsf{contr} : |G| \to \mathcal{R}_{\mathcal{A}}$ defined by

$$\mathsf{contr}(\langle y \rangle) = \langle \bullet \rangle \quad \text{for } y \in G_0$$

$$\mathsf{contr}(\langle e, s \rangle) = \begin{cases} \langle \bullet \rangle & \text{if } e \in T_1 \\ \langle e, s \rangle & \text{if } e \in \mathcal{A} \end{cases} \quad \text{for } e \in G_1,\ s \in [0, 1]$$

Show that contr is well defined and continuous.

b) Show that there is a commutative square

$$\begin{array}{ccc} \pi_1(G, x) & \xrightarrow{\quad \Phi \quad} & \pi_1(|G|, \langle x \rangle) \\ \Big\downarrow{\scriptstyle \Psi^{-1}} & & \Big\downarrow{\scriptstyle \pi_1(\mathsf{contr}, x)} \\ \mathbb{F}_{\mathcal{A}} & \xleftarrow[\quad \mathsf{hol} \quad]{} & \pi_1(\mathcal{R}_{\mathcal{A}}, \langle \bullet \rangle) \end{array}$$

where Φ is the morphism constructed in question 5, and Φ the one constructed in question 15. Deduce that Φ and $\pi_1(\mathsf{contr}, x)$ are group isomorphisms.

17) Consider the infinite grid $\mathbb{G} = (\mathbb{Z} \times \mathbb{R}) \cup (\mathbb{R} \times \mathbb{Z}) \subseteq \mathbb{R}^2$. Compute its fundamental group $\pi_1(\mathbb{G}, (0, 0))$ and give explicit generators for this group.

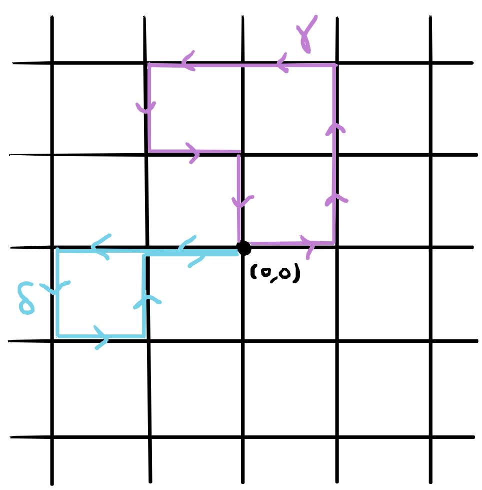

Figure 6: Two loops at $(0, 0)$ in the grid $\mathbb{G}$

[1] Although you proof will certainly generalizes to the infinite case, note that the corresponding statement for infinite graphs is less innocent from a logical point of view, as it requires AC. See for instance this anwser